Multiple Choice

1 B) $\sum\alpha = -1 - 3 + \alpha = -2, \therefore \alpha = 2.$

2 D) Area must be positive. If $f(x) < g(x)$ then

$$\int\left(f(x) - g(x)\right)dx < 0.$$

3 B) Using the pigeon hole principle, 19 from each school plus 1 more from any school, $19 \times 4 + 1 = 77$.

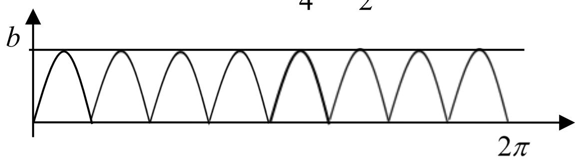
4 A) Domain: $-1 \le 2x \le 1, \therefore -\frac{1}{2} \le x \le \frac{1}{2}.$

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x, \therefore \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}.$$

$$\therefore 2\left(\cos^{-1}2x + \sin^{-1}2x\right) = \pi, \therefore \text{Range: } y = \pi.$$

5 C)

6 D) The period of $\sin 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}.$



7 C) $P(X=4) + P(X=5) = {}^5C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^5 = \frac{1}{64}.$

8 B) For a sample proportion, $\text{Var}(\hat{p}) = \frac{pq}{n} = \frac{\frac{7}{12}\times\frac{5}{12}}{n}$

$$= \frac{35}{144n}.$$

$$\frac{\sqrt{35}}{12\sqrt{n}} < 0.06, \therefore \sqrt{n} > \frac{\sqrt{35}}{12 \times 0.06} = 8.2, \therefore n > 67.5.$$

Note: For a binomial distribution, $\text{Var}(X) = npq$, and for a Bernoulli distribution, $\text{Var}(X) = pq$.

9 A) $\frac{{}^kC_2 + {}^{n-k}C_2}{{}^nC_2} = \frac{\frac{k!}{2!(k-2)!} + \frac{(n-k)!}{2!(n-k-2)!}}{\frac{n!}{2!(n-2)!}}$

$$= \frac{k(k-1) + (n-k)(n-k-1)}{n(n-1)}.$$

10 D) $\alpha = \tan^{-1}\frac{a}{b}, \beta = 2\pi - \tan^{-1}\frac{a}{b}, \gamma = \tan^{-1}\frac{b}{a},$

$$\delta = 2\pi - \tan^{-1}\frac{b}{a}, \therefore \alpha + \beta + \gamma + \delta = 4\pi.$$

Question 11

(a) (i) $2\underset{\sim}{a} - \underset{\sim}{b} = (6\underset{\sim}{i} + 4\underset{\sim}{j}) - (-\underset{\sim}{i} + 4\underset{\sim}{j}) = 7\underset{\sim}{i}.$

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = 3 \times -1 + 2 \times 4 = 5.$

(b) $x^2 - 8x - 9 = (x-9)(x+1) \le 0.$

$-1 \le x \le 9.$

(c) $u = x - 1, du = dx$

$$\int x\sqrt{x-1}\,dx = \int (u+1)\sqrt{u}\,du$$
$$= \int\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$
$$= \frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + C$$
$$= \frac{2}{5}\sqrt{(x-1)^5} + \frac{2}{3}\sqrt{(x-1)^3} + C.$$

(d) $\int\frac{dy}{y} = \int x\,dx \Leftrightarrow \ln y = \frac{x^2}{2} + C.$

$\ln y = \frac{x^2}{2} + C.$

$y = e^{\frac{x^2}{2} + C}.$

(e) $f(x) = \arcsin\left(x^5\right).$

$f'(x) = \frac{5x^4}{\sqrt{1 - x^{10}}}.$

(f) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2.$

$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \frac{1}{4\pi r^2} \times 10 = \frac{5}{2\pi r^2}$ cms^{-1}.

(g) $R = \int_0^{\frac{\pi}{2}}(x - \sin x)dx$

$$= \left[\frac{x^2}{2} + \cos x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi^2}{8} - 1.$$

Question 12

(a) $\begin{pmatrix} a^2 \\ 2 \end{pmatrix} \perp \begin{pmatrix} a+5 \\ a-4 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a^2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} a+5 \\ a-4 \end{pmatrix} = 0.$

$a^3 + 5a^2 + 2a - 8 = 0.$

Let $f(a) = a^3 + 5a^2 + 2a - 8, f(1) = 0, \therefore a-1$ is a factor.

$a^3 + 5a^2 + 2a - 8 = (a-1)(a^2 + 6a + 8)$, by inspection,

$= (a-1)(a+2)(a+4).$

$= 0.$

$\therefore a = -4, -2, 1.$

(b) $V = \pi \int_1^2 x^6 dx = \pi \left[\frac{x^7}{7} \right]_1^2 = \frac{\pi}{7}\left(2^7 - 1\right) = \frac{127\pi}{7}$ u^3.

(c) $p = 0.31, n = 100$

$\text{Var}(\hat{p}) = \frac{pq}{n} = \frac{0.31 \times 0.69}{100} = 0.002139.$

$z = \frac{0.35 - 0.31}{\sqrt{0.002139}} = 0.87.$

$P(X \geq 0.35) = P(z \geq 0.87) = 1 - 0.8078 = 0.1922.$

(d) When $n = 1, 2^{3n} + 13 = 8 + 13 = 21$, which is divisible by 7.

Assume $\exists n \in J: \ 2^{3n} + 13 = 7M, M \in J.$

Required to prove that $2^{3(n+1)} + 13$ is divisible by 7.

$2^{3(n+1)} + 13 = 2^{3n+3} + 13$

$= 8 \times 2^{3n} + 13$

$= 8\left(7M - 13\right) + 13$

$= 7 \times \left(8M - 13\right)$, which is divisible by 7.

By the principle of Induction, $2^{3n} + 13$ is divisible by 7 for all $n \geq 1$.

(e) $\frac{x}{6} \geq \frac{1}{|x-5|} \Leftrightarrow x|x-5| - 6 \geq 0, x \neq 5.$

If $x > 5, x(x-5) - 6 \geq 0.$

$x^2 - 5x - 6 \geq 0.$

$(x+1)(x-6) \geq 0.$

$x \leq -1$ or $x \geq 6.$

Since $x > 5, x \geq 6.$

If $x < 5, x(5-x) - 6 \geq 0.$

$-x^2 + 5x - 6 \geq 0.$

$(x-2)(3-x) \geq 0.$

$2 \leq x \leq 3.$

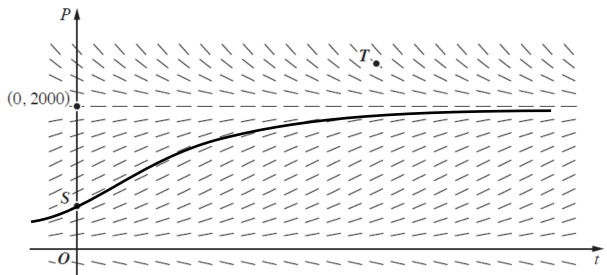
Since $x < 5$, we take $2 \leq x \leq 3.$

$\therefore x \in \left[2, 3\right]$ or $\left[6, \infty\right).$

Question 13

(a) (i) $P = 2000$ is the horizontal asymptote, $\therefore$ any curve below $P = 2000$ must approach this asymptote.

(ii)



(iii) $\frac{dP}{dt} = P(2000 - P)$ is an upside down parabola, meeting the P axis at $P = 0$ and 2000, $\therefore$ maximum when $P = 1000$.

(b) (i) $\cos^4 x + \sin^4 x = \left(\cos^2 x + \sin^2 x\right)^2 - 2\cos^2 x \sin^2 x$

$= 1 - \frac{\left(2\sin x \cos x\right)^2}{2}$

$= 1 - \frac{\sin^2 2x}{2}$

$= 1 - \frac{1}{2}\left(1 - \cos^2 2x\right)$

$= \frac{1 + \cos^2 2x}{2}.$

(ii) $\int_0^{\frac{\pi}{4}} \left(\cos^4 x + \sin^4 x\right) dx = \int_0^{\frac{\pi}{4}} \left(\frac{1 + \cos^2 2x}{2} \right) dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left(1 + \frac{1 + \cos 4x}{2}\right) dx$

$= \left[\frac{3x}{4} + \frac{\sin 4x}{16} \right]_0^{\frac{\pi}{4}}$

$= \frac{3\pi}{16}.$

(c) $\text{proj}_{\underset{\sim}{a}} \underset{\sim}{x} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{x}}{|\underset{\sim}{a}|^2} \right) \underset{\sim}{a} = k\underset{\sim}{a}, \therefore \frac{\underset{\sim}{a} \cdot \underset{\sim}{x}}{|\underset{\sim}{a}|^2} = k.$

$\text{proj}_{\underset{\sim}{b}} \underset{\sim}{x} = \left(\frac{\underset{\sim}{b} \cdot \underset{\sim}{x}}{|\underset{\sim}{b}|^2} \right) \underset{\sim}{b} = p\underset{\sim}{b}, \therefore \frac{\underset{\sim}{b} \cdot \underset{\sim}{x}}{|\underset{\sim}{b}|^2} = p.$

Given $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Let $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$

$\frac{x_1 + 3x_2}{10} = k$ and $\frac{2x_1 - x_2}{5} = p.$

$\therefore x_1 + 3x_2 = 10k.$ (1)

$2x_1 - x_2 = 5p.$ (2)

$(1) + 3 \times (2)$ gives $7x_1 = 10k + 15p, \therefore x_1 = \frac{10k + 15p}{7}.$

$2 \times (1) - (2)$ gives $7x_2 = 20k - 5p, \therefore x_2 = \frac{20k - 5p}{7}.$

$$\therefore \underset{\sim}{x} = \begin{pmatrix} \frac{10k+15p}{7} \\ \frac{20k-5p}{7} \end{pmatrix}.$$

(d) $u = e^x + 2e^{-x}, du = \left(e^x - 2e^{-x}\right)dx.$

$$\int \frac{e^{3x}-2e^x}{4+8e^{2x}+e^{4x}}dx = \int \frac{e^x-2e^{-x}}{4e^{-2x}+8+e^{2x}}dx$$

$$= \int \frac{du}{4+\left(e^x+2e^{-x}\right)^2}$$

$$= \int \frac{du}{4+u^2}$$

$$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2}\tan^{-1}\frac{e^x+2e^{-x}}{2} + C.$$

Question 14

(a) $\frac{dy}{dx} = e^{x+y} = e^x.e^y$

$$\int \frac{dy}{e^y} = \int e^x dx$$

$-e^{-y} = e^x + C.$

When $x = 0, y = 0, -1 = 1 + C, \therefore C = -2.$

$-e^{-y} = e^x - 2.$

$$e^y = \frac{1}{2-e^x}$$

$$y = \ln\frac{1}{2-e^x}$$

$= -\ln(2-e^x).$

Domain: $2 - e^x > 0, \therefore e^x < 2, \therefore x < \ln 2.$

When $x \to -\infty, e^x \to 0, \therefore y \to -\ln 2.$

Range: $y > -\ln 2.$

(b) $f(x) = \frac{kx}{1+x^2} + \arctan x.$

$$f'(x) = \frac{k(1+x^2)-2kx^2}{(1+x^2)^2} + \frac{1}{1+x^2}$$

$$= \frac{k(1-x^2)+(1+x^2)}{(1+x^2)^2}$$

$$= \frac{1+k+x^2(1-k)}{(1+x^2)^2}$$

$= 0$ gives $x^2 = \frac{k+1}{k-1}.$

For $f(x)$ to have an inverse function, $f(x)$ must be monotonic, i.e. $f'(x) \ge 0$ or $f'(x) \le 0$, but not both.

If $\frac{k+1}{k-1} < 0, f'(x) > 0.$

$\frac{k+1}{k-1} < 0 \Leftrightarrow k+1)(k-1) < 0, \therefore -1 < k < 1.$

When $k = 1, f'(x) = \frac{2}{(1+x^2)^2} > 0,$

When $k = -1, f'(x) = \frac{2x^2}{(1+x^2)^2} \ge 0.$

$\therefore$ Inverse function exists for $k \in [-1,1].$

(c) (i) $\frac{d}{dx}\left(\tan^{-1}(3x) + \tan^{-1}(10x)\right) = \frac{3}{1+9x^2} + \frac{10}{1+100x^2} > 0$

for all real $x, \therefore \tan^{-1}(3x) + \tan^{-1}(10x)$ is monotoic increasing. $\therefore$ It has only one real root.

(ii) Let $a = \tan^{-1}(3x), b = \tan^{-1}(10x).$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{3x+10x}{1-(3x)(10x)} = \frac{13x}{1-30x^2}.$$

If $(a+b) = \frac{3\pi}{4}, \tan\frac{3\pi}{4} = -1.$

$$\frac{13x}{1-30x^2} = -1.$$

$\therefore 30x^2 - 13x - 1 = 0.$

$(15x+1)(2x-1) = 0,$

$\therefore x = -\frac{1}{15}$ or $\frac{1}{2}$.

As $\tan^{-1}(3x) + \tan^{-1}(10x) = \frac{3\pi}{4}$, x must be positive,

$\therefore -\frac{1}{15}$ is rejected.

$\therefore$ The only solution is $x = \frac{1}{2}$.

(d) $\underset{\sim}{r}(t) = \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$.

$$D^2(t) = (Vt\cos\theta)^2 + \left(Vt\sin\theta - \frac{1}{2}gt^2\right)^2$$
$$= V^2t^2\left(\cos^2\theta + \sin^2\theta\right) - gV\sin\theta t^3 + \frac{1}{4}g^2t^4$$
$$= V^2t^2 - gV\sin\theta t^3 + \frac{1}{4}g^2t^4.$$

$$\frac{d}{dt}\left(D^2(t)\right) = 2V^2t - 3gV\sin\theta t^2 + g^2t^3$$
$$= t\left(2V^2 - 3gV\sin\theta t + g^2t^2\right).$$

As $t > 0, D^2(t)$ is increasing $\left(\therefore D(t)\right.$ is increasing, as $\left.D(t) > 0\right)$ when $\frac{d}{dt}\left(D^2(t)\right) > 0$.

For $2V^2 - 3gV\sin\theta t + g^2t^2$, it is an upward parabola in terms of t, $\therefore$ the parabola is definite positive when $\Delta < 0$.

$\Delta = 9g^2V^2\sin^2\theta - 8g^2V^2$.

$\Delta < 0$ when $9g^2V^2\sin^2\theta - 8g^2V^2 < 0$.

$\sin^2\theta < \frac{8}{9}$.

$\sin\theta < \sqrt{\frac{8}{9}}$, noting $\sin\theta > 0$.

$\therefore \theta < \sin^{-1}\sqrt{\frac{8}{9}}$.