Multiple Choice

1 C) $(a+ib)^3 = a^3 + 3a^2 ib + 3a(ib)^2 + (ib)^3$
$= a^3 - 3ab^2 + i(3a^2 b - b^3)$

2 D) The converse of "If A then B" is "If B then A"

3 B) $z = \text{cis}\left(-\frac{2\pi}{3}\right), \therefore \bar{z} = \text{cis}\frac{2\pi}{3} = \text{cis}\left(-\frac{4\pi}{3}\right) = z^2$

4 C)

5 A) The direction vectors $\begin{pmatrix}-1\\3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-3\\-1\end{pmatrix}$ indicate that they are paralel, $\therefore$ either they are the same line or they do not intersect.

Put $\lambda = 0$, $\begin{pmatrix}-1\\2\\5\end{pmatrix} = \begin{pmatrix}3\\-10\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\-1\end{pmatrix}, \therefore \begin{cases}-1 = 3+\mu\\ 2 = -10 - 3\mu\\ 5 = 1-\mu\end{cases}$

As $\mu = -4$ satisfies all 3 equations, the 2 lines are the same

6 B) SHM must satisfy $\ddot{x} = -n^2(x-a) \therefore$ equations in the form $A\cos(nt+\alpha) + B\sin(nt+\beta) + C$ are SHM.
In (B), $x = \sin 4t + 4\cos 2t = 2\sin 2t\cos 2t + 4\cos 2t$
$= 2\cos 2t(\sin 2t + 2)$ cannot be converted to the form stated above.

7 A) in (B), $\theta_1 + \theta_2$ can be more or less than the principal range, in (C), $\theta_1 = \theta_2 + 2k\pi$, in (D), $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$, less than the principal range $\therefore$ (A) is correct.

8 D) $|z-i|^2 = x^2 + (y-1)^2, |z-1| = (x-1)^2 + y^2$
If $|z-i|^2 = 4|z-1|^2$ then $x^2 + (y-1)^2 = 4(x-1)^2 + 4y^2$
$\therefore 0 = 3x^2 - 8x + 3y^2 + 2y + 3$. This circle has centre $\left(\frac{4}{3}, -\frac{1}{3}\right)$. Rejected, as it does not match given diagram.
If $|z-1|^2 = 4|z-i|^2$ then $(x-1)^2 + y^2 = 4x^2 + 4(y-1)^2$
$\therefore 0 = 3x^2 + 2x + 3y^2 - 8y + 3$. This circle has centre $\left(-\frac{1}{3}, \frac{4}{3}\right)$.
The shaded region is outside this circle, LHS $<$ RHS
$\therefore$ (D)

9 D) $\underset{\sim}{r} > 0, \underset{\sim}{v} > 0$ (from diagram, tangent to the curve > 0) and $\underset{\sim}{a} < 0$ (since particle is slowing down).

10 B) If A, B and C lie on a sphere of centre at O then $|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}|$. Let it be r.
Then $\underset{\sim}{a}\cdot\underset{\sim}{b} = r^2\cos\alpha = 1, \underset{\sim}{b}\cdot\underset{\sim}{c} = r^2\cos\beta = 2$, and $\underset{\sim}{c}\cdot\underset{\sim}{a} = r^2\cos\gamma = 3$, letting α, β, γ be the corresponding angles.
$\therefore \cos\alpha = \frac{1}{r^2}, \cos\beta = \frac{2}{r^2}, \cos\gamma = \frac{3}{r^2}$.
As long as $r^2 \geq 3$, the 3 equations above have real solutions.

Question 11

(a) $z=\dfrac{3\pm\sqrt{9-4\times4}}{2}=\dfrac{3\pm i\sqrt{7}}{2}=\dfrac{3}{2}\pm i\dfrac{\sqrt{7}}{2}$

(b) $\cos\theta=\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}=\dfrac{1\times-1+2\times4+(-3)\times2}{\sqrt{1^2+2^2+3^2}\sqrt{1^2+4^2+2^2}}$

$=\dfrac{1}{\sqrt{14}\sqrt{21}}$

$\therefore\theta=87°$ to the nearest degree.

(c) $\underset{\sim}{r}=\underset{\sim}{b}+\lambda\overrightarrow{AB}$

$=\underset{\sim}{b}+\lambda(\underset{\sim}{b}-\underset{\sim}{a})$

$=\begin{pmatrix}0\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}0-(-3)\\2-1\\3-5\end{pmatrix}$

$=\begin{pmatrix}0\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$

(d) $\overrightarrow{AB}\,//\,\overrightarrow{DC}$ and $|\overrightarrow{AB}|=|\overrightarrow{DC}|,\therefore\underset{\sim}{b}-\underset{\sim}{a}=\underset{\sim}{c}-\underset{\sim}{d}$

$\overrightarrow{AB}\,//\,\overrightarrow{FE}$ and $|\overrightarrow{AB}|=|\overrightarrow{FE}|,\therefore\underset{\sim}{b}-\underset{\sim}{a}=\underset{\sim}{e}-\underset{\sim}{f}$

$\therefore\underset{\sim}{c}-\underset{\sim}{d}=\underset{\sim}{e}-\underset{\sim}{f}$, i.e. $\overrightarrow{CD}\,//\,\overrightarrow{EF}$ and $|\overrightarrow{CD}|=|\overrightarrow{EF}|$

$\therefore CDFE$ is a parallelogram.

(e) $\ddot{x}=-9(x-4),\therefore n=3$ and centre $x=4$.

$\therefore$ Period $=\dfrac{2\pi}{3}$ and centre $x=4$.

(f) $\displaystyle\int_0^2\frac{5x-3}{(x+1)(x-3)}dx=\int_0^2\left(\frac{A}{x+1}+\frac{B}{x-3}\right)dx$

where $A=\displaystyle\lim_{x\to-1}\frac{5x-3}{x-3}=2, B=\lim_{x\to3}\frac{5x-3}{x+1}=3$

$\therefore I=\displaystyle\int_0^2\left(\frac{2}{x+1}+\frac{3}{x-3}\right)dx$

$=\Big[2\ln|x+1|+3\ln|x-3|\Big]_0^2$

$=2\ln3+3\ln\dfrac{1}{3}$

$=2\ln3-3\ln3$

$=-\ln3$

$=\ln\dfrac{1}{3}$

Question 12

(a) Assume $\sqrt{23}$ is rational, $\sqrt{23}=\dfrac{p}{q}$, where $p,q\in N$

and p and q have no common factors. (1)

$\therefore p^2=23q^2$

$\therefore p^2$ is divisible by 23, i.e. p is divisible by 23. Let $p=23r$

$(23r)^2=23q^2$

$\therefore q^2=23r^2$, i.e. q^2 is divisible by 23, i.e. q is divisible by 23, i.e. p and q have a common factor of 23. (2)

$\therefore$ From (1) and (2), $\sqrt{23}$ is irrational

(b) $(x-y)^2\ge0$

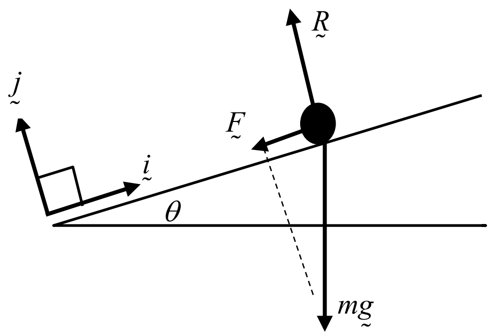
$x^2+y^2\ge2xy$

$2(x^2+y^2)\ge x^2+y^2+2xy$, by adding x^2+y^2 to both sides

$2(x^2+y^2)\ge(x+y)^2$

$\dfrac{(x+y)^2}{x^2+y^2}\le2.$

(c) (i) Resolving the forces, letting F be the resultant force



in the direction of $\underset{\sim}{i}$, $-mg\sin\theta=F$ (1)

in the direction of $\underset{\sim}{j}$, $-mg\cos\theta=R$ (2)

$\therefore\underset{\sim}{F}=-mg\sin\theta\underset{\sim}{i}$

(ii) $m\dfrac{d\underset{\sim}{v}}{dt}=-mg\sin\theta\underset{\sim}{i}$

$\displaystyle\int_0^{\underset{\sim}{v}(t)}d\underset{\sim}{v}=-g\sin\theta\underset{\sim}{i}\int_0^t dt$

$\underset{\sim}{v}(t)=-gt\sin\theta\underset{\sim}{i}.$

(d) $2-2i=2\sqrt{2}e^{-\frac{\pi}{4}+2k\pi}$

$\therefore\sqrt[3]{2-2i}=\sqrt{2}e^{-\frac{\pi}{12}+\frac{2k\pi}{3}}, k=0,1,-1$

$=\sqrt{2}e^{-\frac{\pi}{12}},\sqrt{2}e^{\frac{7\pi}{12}},\sqrt{2}e^{-\frac{3\pi}{4}}.$

(e) (i) because all coefficients are real.

(ii) $(z-2-i)(z-2+i)=(z-2)^2+1=z^2-4z+5.$

$z^4-3z^3+cz^2+dz-30=(z^2-4z+5)(z^2+az+b).$

Equating the constants gives $-30=5b,\therefore b=-6.$

Equating the coefficients of z^3 gives $-3=-4+a,\therefore a=1$

$\therefore z^2+az+b=z^2+z-6=(z+3)(z-2).$

$\therefore$ The other zeros are -3 and 2.

Question 13

(a) $\displaystyle\int \frac{1-x}{\sqrt{5-4x-x^2}}dx = \frac{1}{2}\int\left(\frac{-4-2x}{\sqrt{5-4x-x^2}}+\frac{3}{\sqrt{5-4x-x^2}}\right)dx$

$\displaystyle = \frac{1}{2}\int\left(\frac{-4-2x}{\sqrt{5-4x-x^2}}+\frac{3}{\sqrt{9-(x+2)^2}}\right)dx$

$\displaystyle = \sqrt{5-4x-x^2}+3\sin^{-1}\frac{x+2}{3}+C$

(b) (i) Let $f(k)=k^2-2k-3,\ f'(k)=2k-2\geq 0$ for $k\geq 1$.

$\therefore f(k)$ is increasing for all $k\geq 1$.

When $k=3,\ f(3)=0, \therefore k^2-2k-3\geq 0$ for $k\geq 3$.

(ii) Let $n=3$, LHS $=8$, RHS $=9-2=7$.

$\therefore$ The statement is true for $n=3$.

Assume $2^n\geq n^2-2$ for some integer $n\geq 3$.

RTP $2^{n+1}\geq (n+1)^2-2$.

$$\begin{aligned}\text{LHS} &= 2\times 2^n \geq 2\left(n^2-2\right)\\ &\geq n^2+n^2-4\\ &\geq n^2+2n+3-4, \text{ using part (i), } n^2\geq 2n+3\\ &\geq n^2+2n-1\\ &\geq (n+1)^2-2\\ &= \text{RHS}.\end{aligned}$$

$\therefore$ The statement is true for $n+1$.

$\therefore$ By the principal of Induction, $2^n\geq n^2-2$ for all $n\geq 3$.

(c) (i) $v(0)=\begin{pmatrix}40\cos 30^\circ\\ 40\sin 30^\circ\end{pmatrix}=\begin{pmatrix}20\sqrt{3}\\ 20\end{pmatrix}$.

(ii) $m\underset{\sim}{a}=-4\underset{\sim}{v}-mg=\begin{pmatrix}-4\underset{\sim}{v}_x\\ -4\underset{\sim}{v}_y-m\underset{\sim}{g}\end{pmatrix}$

$\underset{\sim}{a}=\begin{pmatrix}\dfrac{dv_x}{dt}\\ \dfrac{dv_y}{dt}\end{pmatrix}=\begin{pmatrix}-4\underset{\sim}{v}_x\\ -4\underset{\sim}{v}_y-10\end{pmatrix}$, since $m=1, g=10$.

Considering the horizontal dimension,

$\displaystyle\frac{dv_x}{dt}=-4v_x,$

$\displaystyle\int_{20\sqrt{3}}^{v_x(t)}\frac{dv_x}{v_x}=-4\int_0^t dt,$

$\displaystyle\ln\left|\frac{v_x(t)}{20\sqrt{3}}\right|=-4t,$

$v_x(t)=20\sqrt{3}e^{-4t}$

Considering the vertical dimension,

$\displaystyle\frac{dv_y}{dt}=-4\underset{\sim}{v}_y-10,$

$\displaystyle\int_{20}^{v_y(t)}\frac{dv_y}{4v_y+10}=-\int_0^t dt,$

$\displaystyle\therefore \frac{1}{4}\ln\left|\frac{4v_y(t)+10}{90}\right|=-t,$

$4v_y(t)+10=90e^{-4t}$

$\displaystyle v_y(t)=\frac{90e^{-4t}-10}{4}$

$\displaystyle\quad =\frac{45}{2}e^{-4t}-\frac{5}{2}.$

$\therefore v(t)=\begin{pmatrix}20\sqrt{3}e^{-4t}\\ \dfrac{45}{2}e^{-4t}-\dfrac{5}{2}\end{pmatrix}$

(iii) $\displaystyle\frac{dr_x}{dt}=20\sqrt{3}e^{-4t}$

$\displaystyle\int_0^{r_x(t)}dr_x=20\sqrt{3}\int_0^t e^{-4t}dt$

$\displaystyle r_x(t)=20\sqrt{3}\times -\frac{1}{4}\left[e^{-4t}\right]_0^t$

$\quad =5\sqrt{3}\left(1-e^{-4t}\right).$

$\displaystyle\frac{dr_y}{dt}=\frac{45}{2}e^{-4t}-\frac{5}{2}$

$\displaystyle\int_0^{r_y(t)}dr_y=\int_0^t\left(\frac{45}{2}e^{-4t}-\frac{5}{2}\right)dt$

$\displaystyle\quad =\left[-\frac{45}{8}e^{-4t}-\frac{5}{2}t\right]_0^t$

$\displaystyle\quad =\frac{45}{8}\left(1-e^{-4t}\right)-\frac{5t}{2}.$

$\therefore r(t)=\begin{pmatrix}5\sqrt{3}\left(1-e^{-4t}\right)\\ \dfrac{45}{8}\left(1-e^{-4t}\right)-\dfrac{5t}{2}\end{pmatrix}$

(iv) Let $r_y=0, \dfrac{45}{8}\left(1-e^{-4t}\right)-\dfrac{5t}{2}=0$

$\displaystyle 1-e^{-4t}=\frac{8}{45}\times\frac{5t}{2}=\frac{4t}{9}$

From the given diagram, $t\approx 2.25$ sec.

$\therefore r_x=5\sqrt{3}\left(1-e^{-4\times 2.25}\right)=8.7$ m.

Question 14

(a) (i) $z=e^{\frac{i\pi}{6}}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}+i}{2}$,

$w=e^{\frac{3i\pi}{4}}=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=\frac{-\sqrt{2}+i\sqrt{2}}{2}$.

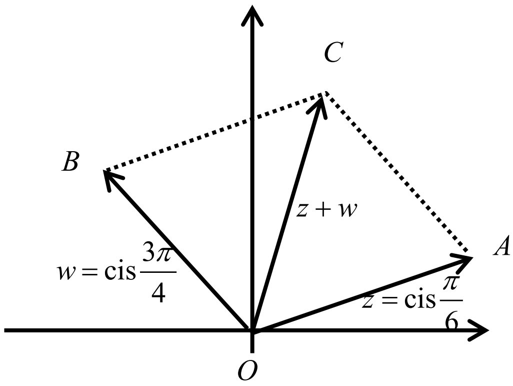
$z+w=\frac{\sqrt{3}-\sqrt{2}+i\left(1+\sqrt{2}\right)}{2}$.

$|z+w|^2=\frac{1}{4}\left(\left(\sqrt{3}-\sqrt{2}\right)^2+\left(1+\sqrt{2}\right)^2\right)$

$=\frac{1}{4}\left(\left(5-2\sqrt{6}\right)+\left(3+2\sqrt{2}\right)\right)$

$=\frac{1}{4}\left(8-2\sqrt{6}+2\sqrt{2}\right)$

$=\frac{1}{2}\left(4-\sqrt{6}+\sqrt{2}\right)$.



(ii) $|z|=|w|=1,\therefore OACB$ is a rhombus, i.e. OC bisects $\angle AOB$.

$\therefore \angle AOC=\frac{1}{2}\angle AOB=\frac{1}{2}\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)=\frac{7\pi}{24}$.

(iii) In $\Delta OAC, \cos\angle AOC=\frac{OA^2+OC^2-AC^2}{2\times OA\times OC}$

$=\frac{OC^2}{2OC}$, since $OABC$ is a rhombus of side $1, OA=AC$

$=\frac{1}{2}OC$

$=\frac{1}{2}\sqrt{\frac{1}{4}\left(8-2\sqrt{6}+2\sqrt{2}\right)}$

$=\frac{1}{4}\sqrt{8-2\sqrt{6}+2\sqrt{2}}$.

(b) The 1st particle has equation $x_1=4\cos\frac{t}{4}$, since period $=8\pi=\frac{2\pi}{n},\therefore n=\frac{1}{4}$, and when $t=0, x=4$.

The 2nd particle has equation $x_2=4\cos\frac{1}{4}\left(t-2\pi\right)$.

They collide when $4\cos\frac{1}{4}\left(t-2\pi\right)=4\cos\frac{1}{4}t$.

$\cos\frac{1}{4}\left(t-2\pi\right)=\cos\frac{1}{4}t$

$\frac{1}{4}\left(t-2\pi\right)=\pm\frac{1}{4}t+2k\pi, k\in J$

$\frac{1}{4}t-\frac{\pi}{2}=-\frac{1}{4}t+2k\pi$

$\frac{1}{2}t=\frac{\pi}{2}+2k\pi$

$t=\pi+4k$.

$\therefore$ They first collide at $t=5\pi$ seconds, ignoring $t=\pi$ because the 2nd particle only moves after $t=2\pi$.

When $t=5\pi, x_1=4\cos\frac{5\pi}{4}=4\times-\frac{\sqrt{2}}{2}=-2\sqrt{2}$.

(c) (i) Let the upward direction be positive,

$Ma=-Mg-kMv^2$

$a=-\left(g+kv^2\right)$.

$\frac{vdv}{dy}=-\left(g+kv^2\right)$

$\int_{v_0}^{0}\frac{vdv}{g+kv^2}=-\int_0^H dy$

$H=\frac{1}{2k}\ln\left|g+kv^2\right|_0^{v_0}$

$=\frac{1}{2k}\ln\frac{g+kv_0^2}{g}$.

(ii) When it returns to the ground, taking the downward direction be positive.

$a=\frac{vdv}{dy}=g-kv^2$

$\int_0^{v_1}\frac{vdv}{g-kv^2}=\int_0^H dy$

$H=-\frac{1}{2k}\ln\left|g-kv^2\right|_0^{v_1}$

$=\frac{1}{2k}\ln\frac{g}{g-kv_1^2}$.

But $H=\frac{1}{2k}\ln\frac{g+kv_0^2}{g},\therefore\frac{g+kv_0^2}{g}=\frac{g}{g-kv_1^2}$.

$g^2=g^2-kgv_1^2+kgv_0^2-k^2v_0^2v_1^2$

$\therefore kv_0^2v_1^2=g\left(v_0^2-v_1^2\right)$

Question 15

(a) (i) Let $u=\sin^{n-1}x, dv=\sin x\,dx$,

$\therefore du=(n-1)\sin^{n-2}x\cos x\,dx$ and $v=-\cos x$.

$$J_n=\left[-\cos x\sin^{n-1}x\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\left(1-\sin^2 x\right)dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^{n}x\,dx$$

$$=(n-1)J_{n-2}-(n-1)J_n.$$

$\therefore nJ_n=(n-1)J_{n-2}$.

$\therefore J_n=\dfrac{n-1}{n}J_{n-2}$.

(ii) Let $x=\sin^2\alpha, dx=2\sin\alpha\cos\alpha\,d\alpha$.

When $x=0,\alpha=0$. When $x=1,\alpha=\dfrac{\pi}{2}$.

$$I_n=\int_0^{\frac{\pi}{2}}\sin^{2n}\alpha\left(1-\sin^2\alpha\right)^n 2\sin\alpha\cos\alpha\,d\alpha$$

$$=2\int_0^{\frac{\pi}{2}}\sin^{2n+1}\alpha\cos^{2n+1}\alpha\,d\alpha$$

$$=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}\left(2\sin\alpha\cos\alpha\right)^{2n+1}d\alpha$$

$$=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}\sin^{2n+1}2\alpha\,d\alpha$$

Let $\alpha=\dfrac{\theta}{2}, d\alpha=\dfrac{1}{2}d\theta$. When $\alpha=\dfrac{\pi}{2},\theta=\pi;\alpha=0,\theta=0$.

$$I_n=\frac{1}{2^{2n}}\times\frac{1}{2}\int_0^{\pi}\sin^{2n+1}\theta\,d\theta$$

$$=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\,d\theta,$$ since the curve $\sin x$ is symmetrical about the line $x=\dfrac{\pi}{2}$, for $0\le x\le\pi$, $\int_0^{\pi}\sin x\,dx=2\int_0^{\frac{\pi}{2}}\sin x\,dx$

(iii) $I_n=\dfrac{1}{2^{2n}}J_{2n+1}$

$$=\frac{1}{2^{2n}}\frac{(2n+1)-1}{2n+1}J_{2n-1},$$ from part (i)

$$=\frac{1}{2^{2n}}\frac{2n}{2n+1}\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\,d\theta,$$ from definition of (i)

$$=\frac{1}{2^{2n}}\frac{2n}{2n+1}\times2^{2(n-1)}I_{n-1},$$ from part (ii), making $\int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\,d\theta$ the subject, then changing n to $n-1$.

$$\therefore I_n=\frac{1}{2^{2n}}\frac{2n}{2n+1}\times\frac{2^{2n}}{4}I_{n-1}$$

$$=\frac{n}{4n+2}I_{n-1}.$$

(b) (i) $\overrightarrow{LP}=\overrightarrow{LA}+\overrightarrow{AC}+\overrightarrow{CP}$

$$=\frac{1}{2}\overrightarrow{BA}+\overrightarrow{AC}+\frac{1}{2}\left(\overrightarrow{CA}+\overrightarrow{AD}\right)$$

$$=-\frac{1}{2}\underset{\sim}{b}+\underset{\sim}{c}+\frac{1}{2}\left(-\underset{\sim}{c}+\underset{\sim}{d}\right)$$

$$=\frac{1}{2}\left(-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}\right)$$

(ii) RHS $=4\left(\left|\overrightarrow{LP}\right|^2+\left|\overrightarrow{MQ}\right|^2+\left|\overrightarrow{NR}\right|^2\right)$

$$=\left|-\underset{\sim}{b}+\underset{\sim}{c}+\underset{\sim}{d}\right|^2+\left|\underset{\sim}{b}-\underset{\sim}{c}+\underset{\sim}{d}\right|^2+\left|\underset{\sim}{b}+\underset{\sim}{c}-\underset{\sim}{d}\right|^2$$

$$=\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2+\left|\underset{\sim}{d}\right|^2+2\left(-\underset{\sim}{b}\cdot\underset{\sim}{c}-\underset{\sim}{b}\cdot\underset{\sim}{d}+\underset{\sim}{c}\cdot\underset{\sim}{d}\right)$$
$$+\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2+\left|\underset{\sim}{d}\right|^2+2\left(-\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{d}-\underset{\sim}{c}\cdot\underset{\sim}{d}\right)$$
$$+\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2+\left|\underset{\sim}{d}\right|^2+2\left(\underset{\sim}{b}\cdot\underset{\sim}{c}-\underset{\sim}{b}\cdot\underset{\sim}{d}-\underset{\sim}{c}\cdot\underset{\sim}{d}\right)$$

$$=3\left(\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2+\left|\underset{\sim}{d}\right|^2\right)-2\left(\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{d}+\underset{\sim}{c}\cdot\underset{\sim}{d}\right)$$

$$=\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2+\left|\underset{\sim}{d}\right|^2+\left(\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2-2\underset{\sim}{b}\cdot\underset{\sim}{c}\right)$$
$$+\left(\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{d}\right|^2-2\underset{\sim}{b}\cdot\underset{\sim}{d}\right)+\left(\left|\underset{\sim}{c}\right|^2+\left|\underset{\sim}{d}\right|^2-2\underset{\sim}{c}\cdot\underset{\sim}{d}\right)$$

$$=\left|\underset{\sim}{b}\right|^2+\left|\underset{\sim}{c}\right|^2+\left|\underset{\sim}{d}\right|^2+\left|\underset{\sim}{b}-\underset{\sim}{c}\right|^2+\left|\underset{\sim}{b}-\underset{\sim}{d}\right|^2+\left|\underset{\sim}{c}-\underset{\sim}{d}\right|^2$$

$$=\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{AC}\right|^2+\left|\overrightarrow{AD}\right|^2+\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{BD}\right|^2+\left|\overrightarrow{CD}\right|^2$$

$=$ LHS.

(c) Students should know that when a particle rotates round a cylinder of radius r, from one end to the other, then its parametric equations could be

$$\begin{cases}x=r\sin nt\\ y=r\cos nt\\ z=t\end{cases}$$

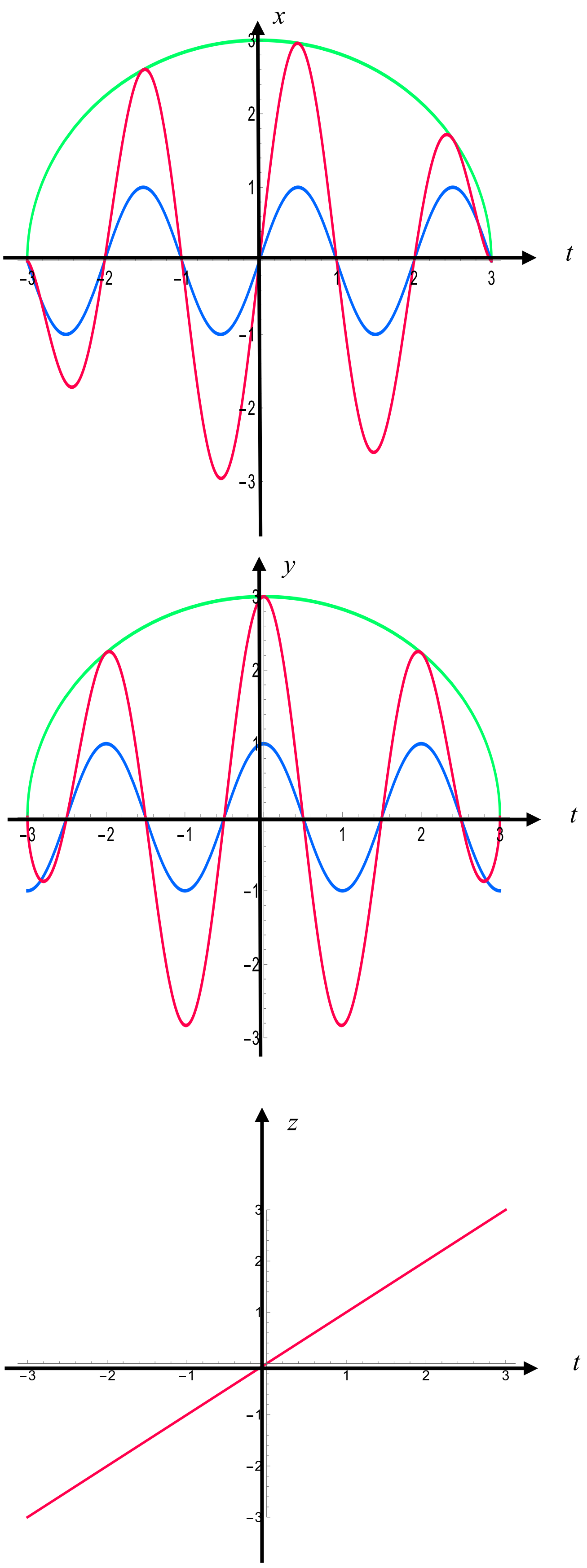
Thus, the parametric equations for the curve ℓ could be

$$\begin{cases}x(t)=\sqrt{9-t^2}\sin(\pi t)\\ y(t)=\sqrt{9-t^2}\cos(\pi t)\\ z(t)=t,-3\le t\le 3\end{cases}$$

as they satisfy the points $(0,0,-3)$ and $(0,0,3)$.

Note:

The diagram below is drawn by Mathematica, showing in red the above parametric equations



Question 16

(a) (i) $w^3 = e^{2i\pi} = 1, \therefore w^3 - 1 = 0.$

But $w^3 - 1 = (w-1)(1+w+w^2)$

$\therefore 1 + w + w^2 = 0,$ since $w \neq 1.$

(ii) $\overrightarrow{BC}$ rot. $120° = \overrightarrow{CA}$

$(c-b)e^{\frac{i2\pi}{3}} = a - c$

$(c-b)w = a - c$

$c(1+w) - bw - a = 0$

$-cw^2 - bw - a = 0,$ since $1 + w + w^2, \therefore 1 + w = -w^2$

$\therefore a + bw + cw^2 = 0$

(ii) If the direction is not important, the result above can be written as $b + cw + aw^2 = 0$ and $c + aw + bw^2 = 0.$

$a^2 = \left(bw + cw^2\right)^2 = b^2w^2 + c^2w + 2bc,$ noting $w^4 = w^3.w = w$

$b^2 = \left(cw + aw^2\right)^2 = c^2w^2 + a^2w + 2ac$

$c^2 = \left(aw + bw^2\right)^2 = a^2w^2 + b^2w + 2ab$

Adding the above 3 lines

$a^2 + b^2 + c^2 = \left(a^2 + b^2 + c^2\right)\left(w + w^2\right) + 2\left(ab + ac + bc\right)$

$a^2 + b^2 + c^2 = -\left(a^2 + b^2 + c^2\right) + 2\left(ab + ac + bc\right)$

$2\left(a^2 + b^2 + c^2\right) = 2\left(ab + ac + bc\right)$

$\therefore a^2 + b^2 + c^2 = ab + ac + bc.$

(b) (i) Let $f(x) = x - \ln x,$ for $x > 0$

$f'(x) = 1 - \frac{1}{x} = 0$ when $x = 1, y = 1$

$f''(x) = \frac{1}{x^2} > 0, \therefore (1,1)$ is a minimum point.

$\therefore f(x) \geq 1.$

$\therefore f(x) > 0$

$\therefore x > \ln x.$

(ii) $1 > \ln 1$

$2 > \ln 2$

$3 > \ln 3$

$\ldots$

$n > \ln n$

$\therefore 1 + 2 + 3 + \ldots + n > \ln\left(1 \times 2 \times 3 \times \ldots \times n\right)$

$\frac{n(n+1)}{2} > \ln(n!)$

$n(n+1) > 2\ln(n!) = \ln(n!)^2$

$\therefore e^{n^2+n} > (n!)^2.$

(c) $|w| = |z| = 1, \therefore \left|\frac{z}{w}\right| = 1.$

Let $u = \frac{xz + yw}{z} = x + y\frac{w}{z}$

$= x + y\,\text{cis}(-\theta)$, as $\frac{\pi}{2} < \theta < \pi$, letting $\text{Arg}\frac{z}{w} = \theta$

$= (x + y\cos\theta) - iy\sin\theta.$

$\therefore$ Since θ lies in the 2nd quadrant, $\sin\theta > 0, \cos\theta < 0.$

For u to lie in the 2nd quadrant, $-y\sin\theta > 0, x + y\cos\theta < 0$

$\therefore y < 0$ (since $\sin\theta > 0$) (1)

and because $y\cos\theta > 0$, in order for $x + y\cos\theta < 0$,

we must have $x < -y\cos\theta$, where $\cos\theta < 0$

$\therefore y > \frac{-x}{\cos\theta}$ (2)

$\therefore$ The region is the area below the line $y = x$ and above the line $y = -\frac{1}{\cos\theta}x$, but due to $\cos\theta < 0$, this line has positive gradient.