## Multiple Choice

1 D) Let $t=0, T=15+4\times 1=19$

2 A) Given $\hat{p}=\frac{9}{12}=\frac{3}{4}, P(\text{at least } 9)=P\left(\hat{p}\geq\frac{3}{4}\right)$

3 C) Join the tangent lines of the solution curve that goes through $(-2,1)$, it goes through $y=1.56$.

4 C) $10+2-3=9$

5 B) $\sin^{-1}(\sin a)=\sin^{-1}(\sin(\pi-a))=\pi-a$

6 C) The projection of $10\underset{\sim}{a}$ on vector $\underset{\sim}{b}$ or $2\underset{\sim}{b}$ is the same, i.e. $10\times$ the projection of $\underset{\sim}{a}, \therefore 10\underset{\sim}{c}$

7 B) $\sin^{-1}x$ is an increasing function, $\therefore$ B

8 A) (A) and (B) are even functions, since they both have $|x|$, i.e. $f(x)=f(-x)$.
Put $x=1$, (A) gives $y=0$, and (B) gives $2, \therefore$ (A)

9 D) If the inverse cuts the line $x=1$ at more than 3 points then its curve cuts the line $y=1$ at more than 3 points. (A), (B) and (C) cut the line $y=1$ at 3 points, only (D) cuts the line $y=1$ at more than 3 points.

10 B) No more than 2 students sit together = 2 students sit together (as it is impossible to sit all students separate) $= {}^5C_2\times 2!\times 3!\times 3!$ (choose 2 students to sit together ${}^5C_2$, allow these 2 students to swap 2!, put 4 groups of 5 students in a circle 3!, put teachers in 3!)

$=\frac{5!}{2!\times 3!}\times 2!\times 3!\times 3!$

$=5!\times 3!$

## Question 11

(a) $t=\frac{x-1}{3}, \therefore y=4\left(\frac{x-1}{3}\right)=\frac{4x}{3}-\frac{4}{3}$.

(b) $\frac{10!}{3!2!}$.

(c) $P(-1)=0, \therefore -1+a-b-12=0, \therefore a-b=13$.

$P(2)=-18, \therefore 8+4a+2b-12=-18, \therefore 2a+b=-7$

$\therefore 3a=6$

$\therefore a=2, b=-11$

(d) $\int\frac{1}{\sqrt{4-9x^2}}dx=\frac{1}{3}\sin^{-1}\frac{3x}{2}+C$

(e) $\cos\theta+\sin\theta=1$

$\sqrt{2}\left(\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}\right)=1$

$\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)=1$

$\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$

$\theta-\frac{\pi}{4}=\pm\frac{\pi}{4}+2k\pi, k\in J$

$\theta=\frac{\pi}{2}+2k\pi, 2k\pi$

For $0\leq\theta\leq 2\pi, \theta=0, \frac{\pi}{2}, 2\pi$

(f) (i) $\operatorname{Var}(\hat{p})=\frac{p(n-p)}{n}=\frac{0.3\times 0.7}{900}=\frac{7}{30000}$

(ii) $z=\frac{0.31-0.3}{\sqrt{\frac{7}{30000}}}=0.655$

$P(\hat{p}\leq 0.31)=P(z\leq 0.66)=0.75$ (to 2 dp)

## Question 12

(a) Let $u = x-3, du = dx$. When $x=3, u=0; x=4, u=1$

$$\int_3^4 (x+2)\sqrt{x-3}\,dx = \int_0^1 (u+5)\sqrt{u}\,du$$

$$= \int_0^1 \left(u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right)du$$

$$= \left[\frac{2u^{\frac{5}{2}}}{5} + 5\times\frac{2u^{\frac{3}{2}}}{3}\right]_0^1$$

$$= \frac{2}{5} + \frac{10}{3} = \frac{56}{15}.$$

(b) Let $n=1$, LHS $= 1\times 2 = 2$, RHS $= 2+0 = 2 =$ LHS.

$\therefore$ The statement is true for $n=1$.

Assume $(1\times 2)+(2\times 2^2)+\ldots+(n\times 2^n) = 2+(n-1)2^{n+1}$

for some positive integer $n$.

RTP $(1\times 2)+(2\times 2^2)+\ldots+(n\times 2^n)+((n+1)\times 2^{n+1})$

$= 2 + n\times 2^{n+2}$.

LHS $= 2+(n-1)2^{n+1} + (n+1)\times 2^{n+1}$

$= 2 + 2n\times 2^{n+1}$

$= 2 + n\times 2^{n+2} =$ RHS.

$\therefore$ The statement is true for $n+1$.

$\therefore$ By the principle of Induction, it's true for all $n \ge 1$.

(c) (i) $\Pr(X=3) = {}^5C_3 \times 0.65^3 \times 0.35^2$

(ii) ${}^5C_3 \times 0.65^3 \times 0.35^2 \times 0.6^4$

(d) ${}^{2022}C_{80} + {}^{2022}C_{81} + {}^{2023}C_{1943} = {}^{2023}C_{81} + {}^{2023}C_{1943}$

$= {}^{2023}C_{81} + {}^{2023}C_{80}$ since ${}^{2023}C_{1943} = {}^{2023}C_{(2023-1943)} = {}^{2023}C_{80}$

$= {}^{2024}C_{81}, \therefore p = 2024, q = 81$.

Accept $p = 2024, q = 2024-81 = 1943$.

(e) $V = V$(cylinder of height 4, radius 10) $+ V$(the region below the curve, $4 \le y \le 12$, rotated about the $y$-axis)

$$= \pi\times 4\times 10^2 + \pi\int_4^{12} x^2\,dy, \text{ where } x = \frac{60}{y} - 5$$

$$= 400\pi + \pi\int_4^{12}\left(\frac{60}{y} - 5\right)^2 dy$$

$$= 400\pi + \pi\int_4^{12}\left(\frac{3600}{y^2} - \frac{600}{y} + 25\right)dy$$

$$= 400\pi + \pi\left[-\frac{3600}{y} - 600\ln y + 25y\right]_4^{12}$$

$$= 400\pi + \pi\left[(-300 - 600\ln 12 + 300) - (-900 - 600\ln 4 + 100)\right]$$

$= 1200\pi - 600\pi\ln 3$ units$^3$.

## Question 13

(a) (i) $\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt}$, where $\dfrac{dV}{dh} = \pi(2Rh - h^2)$

$$= \frac{1}{\pi(2Rh-h^2)}k(2R-h)$$

$$= \frac{k}{\pi h}.$$

(ii) $\dfrac{dh}{dt} = \dfrac{k}{\pi h}$,

$$\frac{\pi}{k}\int_0^R h\,dh = \int_0^T dt.$$

$$T = \frac{\pi}{k}\left[\frac{h^2}{2}\right]_0^R$$

$$= \frac{\pi R^2}{2k}.$$

(iii) $\dfrac{dV}{dt} = k(2R-h) - 2kR$

$= -kh.$

But $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} = \pi(2Rh - h^2)\dfrac{dh}{dt}$.

$\therefore \pi(2Rh-h^2)\dfrac{dh}{dt} = -kh$

$$-\frac{\pi}{k}\int_R^0 (2R-h)\,dh = \int_0^{T_2} dt$$

$$\therefore T_2 = \frac{\pi}{k}\left[2Rh - \frac{h^2}{2}\right]_0^R$$

$$= \frac{\pi}{k}\left(2R^2 - \frac{R^2}{2}\right)$$

$$= \frac{3\pi R^2}{2k}$$

$= 3T$.

(b) (i) $\tan\theta = 2, \therefore \sec^2\theta = 1+\tan^2\theta = 1 + 2^2 = 5$.

$\therefore \cos\theta = \dfrac{1}{\sec\theta} = \dfrac{1}{\sqrt{5}}$.

At the point of collision, $vt\cos\theta = ut$

$\therefore \dfrac{v}{\sqrt{5}} = u, \therefore v = \sqrt{5}u$.

(ii) Let $T$ be the time of collision,

$vT\sin\theta - \frac{1}{2}gT^2 = H - \frac{1}{2}gT^2$

$\therefore H = vT\sin\theta$

$T = \dfrac{H}{v\sin\theta}$, where $v = \sqrt{5}u, \sin\theta = \sqrt{1-\dfrac{1}{5}} = \dfrac{2}{\sqrt{5}}$

$= \dfrac{H}{2u}$.

(iii) $v_A(t)=\begin{pmatrix} v\cos\theta \\ v\sin\theta - gt \end{pmatrix}, v_B(t)=\begin{pmatrix} u \\ -gt \end{pmatrix}$.

The 2 velocity vectors are perpendicular at collision,

$$\therefore \begin{pmatrix} v\cos\theta \\ v\sin\theta - gT \end{pmatrix}\cdot\begin{pmatrix} u \\ -gT \end{pmatrix}=0.$$

$$vu\cos\theta - v\sin\theta gT + (gT)^2 = 0$$

$$\sqrt{5}u^2\times\frac{1}{\sqrt{5}} - \sqrt{5}u\times\frac{2}{\sqrt{5}}g\frac{H}{2u} + g^2\frac{H^2}{4u^2} = 0$$

$$u^2 - gH + \left(\frac{gH}{2u}\right)^2 = 0$$

$$\left(u - \frac{gH}{2u}\right)^2 = 0$$

$$u - \frac{gH}{2u} = 0$$

$$\therefore H = \frac{2u^2}{g}.$$

(iv) At the maximum height, $\dot{y}_A(t)=0$,

$v\sin\theta - gt = 0$,

$$\therefore t = \frac{v}{g}\sin\theta = \frac{\sqrt{5}u}{g}\times\frac{2}{\sqrt{5}} = \frac{2u}{g}.$$

$$\therefore y_A = vt\sin\theta - \frac{1}{2}gt^2$$

$$= \sqrt{5}u\times\frac{2u}{g}\times\frac{2}{\sqrt{5}} - \frac{1}{2}g\frac{4u^2}{g^2}$$

$$= \frac{4u^2}{g} - \frac{2u^2}{g}$$

$$= \frac{2u^2}{g}$$

$$= H.$$

## Question 14

(a) (i) $f'(x) = 2 + \frac{1}{x} > 0$ for $x > 0$,

$\therefore f(x)$ is 1:1, $\therefore$ its inverse is a function.

(ii) $f(1) = 2 + \ln 1 = 2, \therefore g(2) = 1$.

$f'(1) = 2 + 1 = 3$.

$$g'(2) = \frac{1}{f'(1)} = \frac{1}{3}.$$

(b) (i) $(x-c)^2 + \frac{1}{x^2} = c^2$.

$$x^2 - 2cx + \frac{1}{x^2} = 0.$$

$$x^4 - 2cx^3 + 1 = 0. \qquad (1)$$

(ii) The question requires to find $x$ so that the equation $P(x) = x^4 - 2cx^3 + 1 = 0$ has exactly 1 root.

$P'(x) = 4x^3 - 6cx^2$

$= 0$ when $x = 0, 0, \frac{3c}{2}$.

But $P(0)=1, \therefore P(x)$ has exactly 1 root when its curve touches the $x$-axis at $x = \frac{3c}{2}, \therefore P\left(\frac{3c}{2}\right) = 0$.

Sub. to (1) gives $\frac{81c^4}{16} - 2c\times\frac{27c^3}{8} + 1 = 0$

$$\therefore \frac{81c^4}{16} - \frac{108c^4}{16} + 1 = 0$$

$$\therefore \frac{27c^4}{16} = 1$$

$$\therefore c = \sqrt[4]{\frac{16}{27}}, c > 0.$$

(c) (i) Area $(\Delta OAB) = A = \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}|\times\sin\angle AOB$

But $\cos\angle AOB = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$,

$$\therefore \sin\angle AOB = \sqrt{1-\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)^2} = \frac{\sqrt{(|\underset{\sim}{a}||\underset{\sim}{b}|)^2 - (\underset{\sim}{a}.\underset{\sim}{b})^2}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$\therefore A = \frac{1}{2}\sqrt{(|\underset{\sim}{a}||\underset{\sim}{b}|)^2 - (\underset{\sim}{a}\cdot\underset{\sim}{b})^2}$$

$$= \frac{1}{2}\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2}$$

$$= \frac{1}{2}\sqrt{a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2 - a_1^2b_1^2 - a_2^2b_2^2 - 2a_1a_2b_1b_2}$$

$$= \frac{1}{2}\sqrt{a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2}$$

$$= \frac{1}{2}\sqrt{(a_1b_2 - a_2b_1)^2} = \frac{|a_1b_2 - a_2b_1|}{2}$$

(ii) $\overrightarrow{OP}=\overrightarrow{OI}+\overrightarrow{IP}$

$$=\begin{pmatrix} r \\ 0 \end{pmatrix}+\begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$$

$$=\begin{pmatrix} r(1+\cos t) \\ r\sin t \end{pmatrix}$$

$\overrightarrow{OQ}=\overrightarrow{OJ}+\overrightarrow{JQ}$

$$=\begin{pmatrix} -R \\ 0 \end{pmatrix}+\begin{pmatrix} R\cos 2t \\ R\sin 2t \end{pmatrix}$$

$$=\begin{pmatrix} R(-1+\cos 2t) \\ R\sin 2t \end{pmatrix}$$

$$=\begin{pmatrix} R(-2+2\cos^2 t) \\ R\sin 2t \end{pmatrix}$$

Using the result of part (i),Area$(\Delta OPQ)=\frac{1}{2}\left|x_P y_Q - y_P x_Q\right|$

$=\frac{1}{2}\left|rR(1+\cos t)\sin 2t - rR\sin t(-2+2\cos^2 t)\right|$

$=rR\left|(1+\cos t)\sin t\cos t-\sin t(\cos^2 t-1)\right|$

$=rR\left|\sin t\cos t+\sin t\right|$.

Let $f(t)=\sin t\cos t+\sin t$

$f'(t)=\cos^2 t-\sin^2 t+\cos t$

$=\cos^2 t-\left(1-\cos^2 t\right)+\cos t$

$=2\cos^2 t+\cos t-1$

$=\left(2\cos t-1\right)\left(\cos t+1\right)$

$=0$ when $\cos t=\frac{1}{2}$ (Ignore $\cos t=-1$, as Area $=0$)

$\therefore t=\frac{\pi}{3}$.

$f''(t)=-4\cos t\sin t-\sin t$

$=-\sin t\left(4\cos t+1\right)<0$ when $t=\frac{\pi}{3}$.

$\therefore$ The area is largest when $t=\frac{\pi}{3}$ or $-\frac{\pi}{3}$ (due to symmetry).