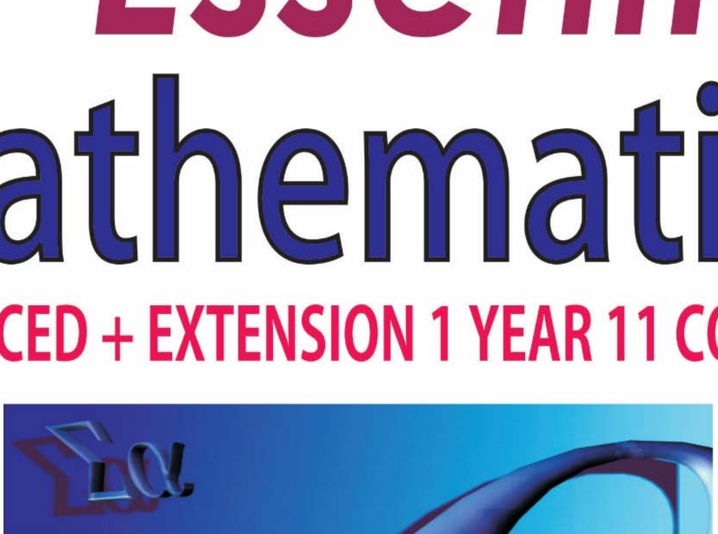
New Essential Mathematics

ADVANCED + EXTENSION 1 YEAR 11 COURSES



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NEW ESSENTIAL MATHEMATICS

ADVANCED + EXTENSION 1 YEAR 11 COURSES

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Preface

This combined book is written for the new Mathematics Advanced + Extension 1 year 11 courses, which are being introduced into the NSW syllabus in 2020.

This book has been written with two main objectives: it can be used as a textbook for classroom use, as well as a step-by-step resource to be used independently by students for their own self-study purposes. This book provides sufficiently clear explanations about each topic in the syllabus, with worked out examples and alternative methods, where applicable. Questions are categorised by topic and graded from easy to hard, to help guide students in their learning. Each chapter also contains a set of review exercises, as well as fully worked solutions for each question. The review exercises will help consolidate students' skills and knowledge, while improving their competence and confidence. The book also features many challenging but feasible problems, usually placed at the end of each exercise as well as in the review exercises. While they may extend beyond the syllabus, they are designed to provide extra stimulus for highly motivated students and increase confidence for the harder questions in the Higher School Certificate examination.

This book also features colour-coding throughout to highlight various theorems and study tips – this makes the book a study reference and more enjoyable to read. Students are advised to complete as many questions in this book as possible to master the course.

This book builds upon what the Terry Lee series has been famous for: it includes many fully explained tips and tricks to help students understand and solve problems efficiently, while ultimately developing a greater enjoyment of the course.

Terry Lee

1 Functions

HSC Outcomes
A student
uses algebraic and graphical techniques to solve, and where appropriate, compare alternative solutions to problems
uses the concepts of functions and relations to model, analyse and solve practical problems
uses appropriate technology to investigate, organise, model and interpret information in a range of contexts
provides reasoning to support conclusions which are appropriate to the context
manipulates algebraic expressions and graphical functions to solve problems
communicates making comprehensive use of mathematical language, notation, diagrams and graph

In this chapter,

One of the most important aspects of any science is the establishment of relationships among variables. For example, a medical researcher would like to investigate the effect of a new drug in blood; an educational psychologist would like to study the relationship between teaching time and learning time, etc. This chapter deals with common functions defined in terms of real numbers.

1.1 Real numbers

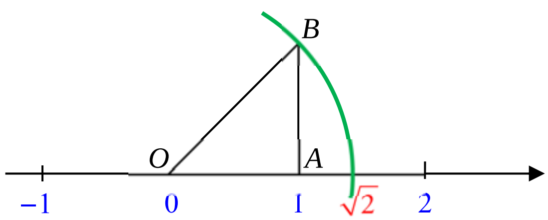
1.1.1 The real number line

Real numbers are a mathematical abstraction commonly used when modelling real phenomenon. Real numbers include all rational numbers plus numbers that cannot be written as rational numbers such as $\sqrt{2}, \pi, e$, etc. A real number may be represented by a point on the real number line.

Example 1.1

Plot $\sqrt{2}$ in its exact place on the real number line.

By Pythagoras $\sqrt{2} = \sqrt{1^2 + 1^2}$, we can plot $\sqrt{2}$ this way: First draw an isosceles right-angled triangle *OAB*, where *O* and *A* are points 0 and 1 respectively. The arc with centre *O*, radius *OB* would cut the number line at $\sqrt{2}$.



1.1.2 Index laws

$$a^m \times a^n = a^{m+n}.$$

$$\frac{a^m}{a^n} = a^{m-n}.$$

$$\left(a^m\right)^n = a^{mn}.$$

Example 1.2

Simplify

(a) $\frac{3^{-2}}{6^{-3}}$. (b) $\frac{3^4 \times 6^{-3}}{9}$. (c) $\frac{2^n}{4^{n+1}}$. (d) $\frac{\left(2^n\right)^2}{4^{n^2}}$.

(a) $\frac{3^{-2}}{6^{-3}} = \frac{6^3}{3^2} = \frac{3^3 \times 2^3}{3^2} = 3 \times 2^3 = 3 \times 8 = 24$.

(b) $\frac{3^4 \times 6^{-3}}{9} = \frac{3^4}{3^2 \times 3^3 \times 2^3} = \frac{3^4}{3^5 \times 2^3} = \frac{1}{3 \times 2^3} = \frac{1}{24}$.

(c) $\frac{2^n}{4^{n+1}} = \frac{2^n}{\left(2^2\right)^{n+1}} = \frac{2^n}{2^{2n+2}} = \frac{2^n}{2^{2n} \times 4} = \frac{1}{4 \times 2^n}$.

(d) $\frac{\left(2^n\right)^2}{4^{n^2}} = \frac{2^{2n}}{4^{n^2}} = \frac{4^n}{4^{n^2}} = 4^{n-n^2}$.

1.1.3 Quadratic surds

Example 1.3

(a) Simplify

(i) $\sqrt{8}+\sqrt{18}$ (ii) $\dfrac{1}{\sqrt{5}+2}$.

(b) Prove that $\dfrac{4}{2+\sqrt{5}}-\dfrac{1}{9-4\sqrt{5}}$ is a rational number.

(a) (i) $\sqrt{8}+\sqrt{18}=2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$.

(ii) $\dfrac{1}{\sqrt{5}+2}=\dfrac{\sqrt{5}-2}{\left(\sqrt{5}+2\right)\left(\sqrt{5}-2\right)}=\dfrac{\sqrt{5}-2}{5-4}=\sqrt{5}-2$.

(b)
$$\begin{aligned}\frac{4}{2+\sqrt{5}}-\frac{1}{9-4\sqrt{5}}&=\frac{4\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}-\frac{9+4\sqrt{5}}{\left(9-4\sqrt{5}\right)\left(9+4\sqrt{5}\right)}\\&=\frac{4\left(2-\sqrt{5}\right)}{4-5}-\frac{9+4\sqrt{5}}{81-80}\\&=-4\left(2-\sqrt{5}\right)-\left(9+4\sqrt{5}\right)\\&=-8+4\sqrt{5}-9-4\sqrt{5}\\&=-17,\ \text{which is a rational number.}\end{aligned}$$

Exercise 1.1

1 Plot $\sqrt{3}$ and $\sqrt{5}$ in their exact positions on the real number line.

2 Simplify the following. Leave your answers in positive index form or surd form.

(a) $x^3\times x^{-3}$. (b) $\dfrac{x^3}{x^{-3}}$. (c) $\left(x^3\times x^4\right)^2$. (d) $\left(\sqrt[3]{x}\right)^{\frac{2}{3}}$.

(e) $\dfrac{(8a)^{-1}}{2a}$. (f) $\dfrac{(9a)^{-3}}{3a^4}$. (g) $\dfrac{(16a)^{-\frac{1}{4}}}{\left(4a^2\right)^{-\frac{1}{2}}}$. (h) $\dfrac{\left(81a^2\right)^{\frac{3}{4}}}{\left(64a^6\right)^{-\frac{2}{3}}}$.

(i) $\dfrac{2^n}{8^n}$. (j) $\dfrac{2^{n+1}}{4^{n-1}}$. (k) $\dfrac{2^{n-1}\times 9^{n-2}}{6^{n+2}}$. (l) $\dfrac{12^{n-2}}{4^{n-1}\times 3^{n+2}}$.

(m) $\dfrac{2\times 4^{n+1}}{4\times 2^{n-1}}$. (n) $\dfrac{2^n\times 3^{n-2}}{6^{2-n}}$. (o) $\dfrac{4^{n-1}\times 18^n}{\left(6^{n+1}\right)^2}$. (p) $\dfrac{10\times 25^{n-4}}{5^{n+2}}$.

3 By raising the following to a common power, arrange the following in ascending order.

(a) $\sqrt{2},\sqrt[4]{3},\sqrt[6]{6}$. (b) $\sqrt{2},\sqrt[3]{3},\sqrt[5]{5}$. (c) $2^{\frac{1}{2}}\times 5^{\frac{2}{3}},3^{\frac{1}{2}}\times 4^{\frac{2}{3}}$. (d) $\left(\dfrac{5}{3}\right)^{\frac{4}{5}},\left(\dfrac{4}{3}\right)^{\frac{3}{5}},\left(\dfrac{4}{3}\right)^{\frac{2}{5}}$.

4 Simplify, where possible.

(a) $\sqrt{8}+\sqrt{50}$. (b) $\sqrt{90}-\sqrt{40}$. (c) $\sqrt{75}-\sqrt{27}$. (d) $\sqrt{18}-\sqrt{12}$.

(e) $\sqrt{12}+\sqrt{48}-\sqrt{98}$. (f) $\sqrt{98}+\sqrt{200}-\sqrt{20}$. (g) $3\sqrt{5}+\sqrt{10}+\sqrt{20}$. (h) $2\sqrt{3}+3\sqrt{2}-\sqrt{108}$.

(i) $\left(\sqrt{3}-\sqrt{2}\right)\sqrt{6}$. (j) $\sqrt{3}\left(\sqrt{8}-\sqrt{6}\right)$. (k) $\sqrt{2}\left(\sqrt{6}+2\sqrt{10}\right)$. (l) $\sqrt{5}\left(\sqrt{15}-\sqrt{5}\right)$.

(m) $\dfrac{\sqrt{10}+\sqrt{40}}{\sqrt{2}}$. (n) $\dfrac{\sqrt{24}+\sqrt{54}}{\sqrt{3}}$. (o) $\dfrac{4+\sqrt{12}}{2}$. (p) $\dfrac{3-\sqrt{27}}{6}$.

(q) $\left(2\sqrt{3}+\sqrt{2}\right)^2$. (r) $\left(\sqrt{5}-2\sqrt{2}\right)^2$. (s) $\left(3\sqrt{6}+2\right)^2$. (t) $\left(2\sqrt{2}-3\sqrt{3}\right)^2$.

5 Express with rational denominators, then simplify, where possible.

(a) $\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{2}}$. (b) $\dfrac{1}{\sqrt{5}}-\dfrac{1}{\sqrt{2}}$. (c) $\dfrac{1}{2\sqrt{5}}+\dfrac{1}{3\sqrt{2}}$. (d) $\sqrt{\dfrac{2}{3}}+\sqrt{\dfrac{2}{5}}$.

(e) $\dfrac{3}{\sqrt{3}+\sqrt{2}}$. (f) $\dfrac{\sqrt{5}}{\sqrt{5}+1}$. (g) $\dfrac{\sqrt{6}}{\sqrt{6}+\sqrt{2}}$. (h) $\dfrac{2\sqrt{3}}{\sqrt{5}-\sqrt{2}}$.

(i) $\dfrac{1}{3\sqrt{2}+1}$. (j) $\dfrac{\sqrt{8}}{2\sqrt{3}+\sqrt{2}}$. (k) $\dfrac{\sqrt{2}+1}{\sqrt{3}-\sqrt{2}}$. (l) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.

(m) $\dfrac{1}{\sqrt{2}+1}+\dfrac{1}{\sqrt{3}+1}$. (n) $\dfrac{1}{2+\sqrt{3}}+\dfrac{\sqrt{2}}{1+\sqrt{2}}$. (o) $\dfrac{1}{\sqrt{3}+\sqrt{5}}-\dfrac{\sqrt{5}}{\sqrt{3}-\sqrt{5}}$. (p) $\dfrac{\sqrt{3}}{\sqrt{3}+1}+\dfrac{1}{\sqrt{2}-1}$.

(q) $\dfrac{\sqrt{3}-2}{\sqrt{3}+1}-\dfrac{\sqrt{3}-1}{\sqrt{3}+2}$. (r) $\dfrac{1}{\left(\sqrt{6}-\sqrt{5}\right)^2}$. (s) $\dfrac{\sqrt{5}}{\left(\sqrt{5}-2\right)^2}$. (t) $\left(\dfrac{\sqrt{2}+1}{\sqrt{3}+1}\right)^2$.

6 Prove that the following are rational numbers.

(a) $\dfrac{2}{3+\sqrt{2}}+\dfrac{1}{2\sqrt{2}-1}$. (b) $\dfrac{2}{3-\sqrt{3}}+\dfrac{11}{6+\sqrt{3}}$. (c) $\dfrac{3}{\sqrt{3}+2}+\dfrac{6}{\sqrt{3}-1}$. (d) $\dfrac{1}{4+\sqrt{5}}-\dfrac{4}{\sqrt{5}+7}$.

(e) $a+\dfrac{1}{a}$, where $a=\dfrac{3-\sqrt{2}}{3+\sqrt{2}}$. (f) $a^2+\dfrac{1}{a^2}$, where $a=1-\sqrt{2}$.

7 Evaluate the following.

(a) $\dfrac{x-1}{x+1}$, where $x=\dfrac{1}{\sqrt{2}-1}$. (b) $\dfrac{1-x^2}{1+x^2}$, where $x=\dfrac{1}{1+\sqrt{2}}$.

1.2 Quadratic equations

Example 1.4

Solve the following quadratic equations by using all available methods, where possible: factorising, quadratic formula and completing the square.

(a) $x^2-x-6=0$. (b) $x^2-4x-3=0$. (c) $2x^2-x-6=0$.

(a) $x^2-x-6=0$.

By factorising,

$x^2-x-6=(x-3)(x+2)=0$.

$\therefore x=-2$ or 3.

By quadratic formula, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $a=1, b=-1, c=-6$

$$x=\frac{1\pm\sqrt{1-4(1)(-6)}}{2}=\frac{1\pm5}{2}=-2 \text{ or } 3.$$

By completing the square,

$x^2 - x = 6.$

$x^2 - x + \frac{1}{4} = 6 + \frac{1}{4}$, since $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$, where here $b = -1$.

$\left(x - \frac{1}{2}\right)^2 = \frac{25}{4}.$

$x - \frac{1}{2} = \pm\frac{5}{2}.$

$x = -2$ or 3.

(b) $x^2 - 4x - 3 = 0$.

By factorising: impossible.

By quadratic formula,

$$x = \frac{4 \pm \sqrt{16+12}}{2}$$
$$= 2 \pm \sqrt{7}$$
$$\simeq -0.65 \text{ or } 4.65.$$

By completing the square,

$x^2 - 4x = 3.$

$x^2 - 4x + 4 = 3 + 4.$

$(x-2)^2 = 7.$

$x - 2 = \pm\sqrt{7}.$

$x = 2 \pm \sqrt{7}.$

(c) $2x^2 - x - 6 = 0.$

By factorising,

$2x^2 - x - 6 = (2x+3)(x-2) = 0.$

$\therefore x = 2$ or $-\frac{3}{2}.$

By quadratic formula,

$$x = \frac{1 \pm \sqrt{1 - 4(2)(-6)}}{2 \times 2}$$
$$= \frac{1 \pm 7}{4}$$
$$= 2 \text{ or } -\frac{3}{2}.$$

By completing the square,

$x^2 - \frac{1}{2}x = 3.$

$x^2 - \frac{1}{2}x + \frac{1}{16} = 3 + \frac{1}{16}.$

$\left(x - \frac{1}{4}\right)^2 = \frac{49}{16}.$

$x - \frac{1}{4} = \pm\frac{7}{4}.$

$x = 2$ or $-\frac{3}{2}.$

Exercise 1.2

1 Solve the following quadratic equations by using all available methods, where possible: (α) factorising, (β) quadratic formula and (γ) completing the square.

(a) $x^2 + 2x - 3 = 0$. (b) $x^2 + 6x + 9 = 0$. (c) $x^2 - 6x + 5 = 0$. (d) $3x^2 - 12x - 36 = 0$.

(e) $3x^2 + 4x - 6 = 0$. (f) $x^2 + 3x - 1 = 0$. (g) $x^2 + x - 4 = 0$. (h) $3x^2 + 2x - 4 = 0$.

(i) $3x^2-5x+2=0$. (j) $5+3x-2x^2=0$. (k) $\frac{1}{3}x^2+\frac{1}{2}x-6=0$.

1.3 Equations reducible to quadratics

Example 1.5

Solve the following equations

(a) $\left(x+\frac{4}{x}\right)^2-4\left(x+\frac{4}{x}\right)-5=0$. (b) $2^{2x+1}+2^x-1=0$. (c) $\frac{x^2+2}{x}-\frac{12x}{x^2+2}=1$.

(a) Let $u=x+\frac{4}{x}$, then the equation becomes $u^2-4u-5=0$.

$(u-5)(u+1)=0.$

$u=5$ or -1.

If $x+\frac{4}{x}=5$ then $x^2-5x+4=0.$

$(x-4)(x-1)=0.$

$\therefore x=4,1.$

If $x+\frac{4}{x}=-1$ then $x^2+x+4=0$. This equation has no real solution.

$\therefore$ Conclusion: $x=4$ or 1.

(b) Let $u=2^x$, noting $2^{2x+1}=2\times 2^{2x}=2\left(2^x\right)^2=2u^2$, then the equation becomes $2u^2+u-1=0$.

$(2u-1)(u+1)=0.$

$u=\frac{1}{2}$ or -1.

If $2^x=\frac{1}{2}=2^{-1}$ then $x=-1$.

The equation $2^x=-1$ has no solution because $2^x>0$ always.

Conclusion: $x=-1$.

(c) Let $u=\frac{x^2+2}{x}$ then the equation becomes $u-\frac{12}{u}=1$.

$u^2-u-12=0.$

$(u-4)(u+3)=0.$

$u=4$ or -3.

If $\frac{x^2+2}{x}=4$ then $x^2-4x+2=0.$

$$x=\frac{4\pm\sqrt{8}}{2}=2\pm\sqrt{2}.$$

If $\frac{x^2+2}{x}=-3$ then $x^2+3x+2=0.$

$(x+1)(x+2)=0.$

$x=-1,-2.$

Conclusion: $x=-1,-2,2\pm\sqrt{2}$.

Example 1.6

John and Sally working together can do a job in 12 days. It takes John 5 days longer than Sally to do the job when each works alone. How many days will it take John to do the job by himself?

Let n be the number of days that John takes to complete the job by himself, then $n – 5$ is the number of days that Sally takes to do her job.

In one day, John does $\frac{1}{n}$ of the job, and Sally does $\frac{1}{n-5}$ of the job, and when working together John and Sally do $\frac{1}{12}$ of the job.

$\frac{1}{n}+\frac{1}{n-5}=\frac{1}{12}$.

$12(n-5)+12n=n(n-5)$.

$12n-60+12n=n^2-5n$.

$n^2-29n+60=0$.

$n=\frac{29\pm\sqrt{29^2-240}}{2}=26.8$ or 2.2.

$\therefore n=26.8$ (2.2 is rejected because $n>5$)

$\therefore$ It will take John 26.8 days to complete the job by himself.

Note: This method is called the unitary method, i.e. find the portion of the task to be completed in 1 day.

Exercise 1.3

1 Solve the following equations for x.

(a) $2\left(\frac{x+2}{x}\right)^2-\frac{x+2}{x}=1$.

(b) $2\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)-5=0$.

(c) $\left(x^2+1\right)^2-\left(x^2+1\right)-6=0$.

(d) $\left(2x^2-x\right)^2-2\left(2x^2-x\right)-3=0$.

(e) $x^2-3x=\frac{36}{x^2-3x}+16$.

(f) $\frac{x^2}{2x-3}-\frac{6x-9}{x^2}-2=0$.

(g) $4\times3^{2x}+3^{x+1}-1=0$.

(h) $16^x-7\times4^x-18=0$.

(i) $\left(x^2+1\right)^4-3\left(x^2+1\right)^2-54=0$.

(j) $16\left(\frac{x}{x+1}\right)^4-73\left(\frac{x}{x+1}\right)^2+36=0$.

(k) $x^2-6x-5=\sqrt{x^2-6x-3}$.

(l) $2x^2-10x=\sqrt{33x^2-165x-54}$.

2 (a) A swimming pool can be filled by the smaller pipe in 7 hours and by the larger pipe in 3 hours. How long does it take to fill the pool by using both pipes?

(b) The hot-water tap takes 12 minutes longer than the cold water tap to fill a bath tub. Both taps, when running together, takes 4 minutes. How long does the hot water tap take to fill the tub by itself?

3 A spiral staircase turns 270° as it rises 3 metres. The radius of the staircase is 1 m, how long is its outside hand railing?