Multiple Choice

1 A) $1 < x \le 3$ and $30° \le \theta < 60°$.

2 D) Line 4, as $(a+2)^2 = 2^2 \Leftrightarrow a+2 = \pm 2$.

3 B) If $\overrightarrow{AP} \ne \lambda \overrightarrow{AB}$ then P is not on the line $AB, \forall \lambda \in R$.

4 C) $x^3 - x^2 + x - 1 = x^2(x-1) + (x-1) = (x-1)(x^2+1)$,

$\therefore \frac{x-1}{(x-1)(x^2+1)} = \frac{1}{x^2+1}, x \ne 1,$

$\therefore \int \frac{1}{x^2+1} dx = \tan^{-1} x$, not $\log x$.

5 C) $\int_a^x f(t)dt = g(x) \Leftrightarrow \frac{d}{dx}\left(g(x)\right) = f(x)$,

$\therefore \int f(x)g(x)dx = \int u du = \frac{1}{2}u^2$, where $u = g(x)$

$\therefore \int f(x)g(x)dx = \frac{1}{2}\left[g(x)\right]^2$.

6 A) Let $z = x + iy, \therefore \bar{z} = x - iy$ and $iz = -y + ix$.

Then $\bar{z} = iz$ when $x = -y$.

Note that (B) is impossible as $|z^2|$ is real, (C) is impossible as $-y \ne y$ and in (D), $\arg(z^3) = 3\arg(z)$.

7 D) The statement is false, e.g. 3 is prime but $\frac{3 \times 4}{2} = 6$ is not prime.

The converse is: If $\frac{n(n+1)}{2}$ is a prime number then n is a prime number.

As $n(n+1)$ is the product of 2 consecutive numbers, $\therefore$ it's always even. $\therefore \frac{n(n+1)}{2}$ is always even for $n \ge 3$, i.e. it is not a prime number.

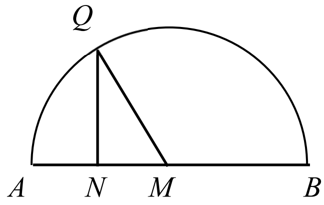
$\therefore \frac{n(n+1)}{2}$ is prime only when $n = 2$, which is a prime number.

$\therefore$ The converse is true.

8 B) $\begin{pmatrix} 0 \\ -mg \end{pmatrix} - kv\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$, as $-mg - kv^2 = ma$,

where $v = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ and $a = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$.

9 D) S_1 is a sphere of diameter AB and S_2 is the perpendicular bisector plane of AM. $\therefore$ The intersection is a circle of centre N at the midpoint of AM, radius NQ.



$$NQ^2 = MQ^2 - NM^2$$
$$= \left(\frac{1}{2}AB\right)^2 - \left(\frac{1}{4}AB\right)^2$$
$$= \frac{3}{16}AB^2.$$
$$\therefore NQ = \frac{\sqrt{3}}{4}AB.$$

10 C) If the particle is initially moving downwards,

$ma = mg - mkv$

$a = \frac{dv}{dt} = g - kv.$

Terminal velocity occurs when $a = 0, \therefore v = \frac{g}{k}$.

If its initial velocity $< \frac{g}{k}$, it will increase until $= \frac{g}{k}$.

If its initial velocity $> \frac{g}{k}$, it will decrease until $= \frac{g}{k}$.

$\therefore$ Both (A) and (B) are wrong.

If the particle is initially moving upwards,

$ma = -mg - mkv$

$a = \frac{dv}{dt} = -(g + kv)$

$\int_U^v \frac{dv}{kv+g} = -\int_0^t dt$, letting U be the initial speed

$\frac{1}{k}\ln\left|\frac{kv+g}{kU+g}\right| = -t.$

$\frac{kv+g}{kU+g} = \pm e^{-kt}.$

$v = \frac{1}{k}\left(\pm(kU+g)e^{-kt} - g\right).$

$\therefore$ As $t \to \infty, v \to -\frac{g}{k}$ = the terminal speed. This negative sign means the particle will reach the maximum height, where $v = 0$, then return to the ground, eventually approach a terminal speed $= \frac{g}{k}$.

Question 11

(a) $\frac{3-i}{2+i}=\frac{3-i}{2+i}\times\frac{2-i}{2-i}=\frac{5-5i}{5}=1-i.$

(b) Let $u=\sin 2x, du=2\cos 2x dx.$

$$\int \sin^3 2x\cos 2x dx=\frac{1}{2}\int u^3 du$$
$$=\frac{u^4}{8}+C$$
$$=\frac{1}{8}\sin^4 2x+C.$$

(c) (i) $-\sqrt{3}+i=2\operatorname{cis}\frac{5\pi}{6}=2e^{\frac{i5\pi}{6}}.$

(ii) $\left(-\sqrt{3}+i\right)^{10}=2^{10}e^{\frac{i25\pi}{3}}$
$$=1024e^{i\left(\frac{\pi}{3}+4\times 2\pi\right)}$$
$$=1024e^{\frac{i\pi}{3}}$$
$$=1024\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$
$$=512+i512\sqrt{3}.$$

(d) $\overrightarrow{BA}=\underset{\sim}{a}-\underset{\sim}{b}=\left(\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}\right)-\left(2\underset{\sim}{j}-\underset{\sim}{k}\right)=\underset{\sim}{i}-3\underset{\sim}{j}+3\underset{\sim}{k}$

$\overrightarrow{BC}=\underset{\sim}{c}-\underset{\sim}{b}=\left(2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)-\left(2\underset{\sim}{j}-\underset{\sim}{k}\right)=2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}.$

$\cos\angle ABC=\frac{\overrightarrow{BA}.\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|}=\frac{2+3+6}{\sqrt{1+9+9}\sqrt{4+1+4}}=\frac{11}{\sqrt{171}}.$

$\therefore \angle ABC=33°,$ to the nearest degree.

(e) $\ell_2:\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-6\\5\end{pmatrix}+\mu\begin{pmatrix}3\\2\end{pmatrix}.$

$$\begin{cases}x=-6+3\mu & (1)\\ y=5+2\mu & (2)\end{cases}$$

$(1)\times 2-(2)\times 3$ gives $2x-3y=-27.$

$\therefore \ell_2: y=\frac{2}{3}x+9.$

(f) Let $t=\tan\frac{x}{2},$

$dt=\frac{1}{2}\sec^2\frac{x}{2}dx=\frac{t^2+1}{2}dx, \therefore dx.=\frac{2dt}{1+t^2}.$

$$\int\frac{dx}{1+\cos x-\sin x}=\int\frac{1}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}\frac{2dt}{1+t^2}$$
$$=\int\frac{dt}{1-t}$$
$$=\ln|1-t|+C$$
$$=\ln\left|1-\tan\frac{x}{2}\right|+C.$$

Question 12

(a) $(\sqrt{a}-\sqrt{b})^2\ge 0.$

$a+b-2\sqrt{ab}\ge 0.$

$\frac{a+b}{2}\ge\sqrt{ab}.$

(b) $a=\frac{dv}{dt}=12-6t.$

$$\int_0^v dv=\int_0^t(12-6t)dt$$

$v=12t-3t^2.$

Maximum velocity occurs when $a=0, \therefore t=2.$

$\int_0^x dx=\int_0^2(12t-3t^2)dt,$ since $v=\frac{dx}{dt}.$

$x=\left[6t^2-t^3\right]_0^2=24-8=16.$

(c) (i) $ma=v\frac{dv}{dx}=-\left(v+3v^2\right),$ as $m=1, a=v\frac{dv}{dx}.$

$\frac{dv}{dx}=-(1+3v)$

(ii) $\int_u^v\frac{dv}{1+3v}=-\int_0^x dx$

$\frac{1}{3}\ln\left|\frac{1+3v}{1+3u}\right|=-x.$

$x=\frac{1}{3}\ln\frac{1+3u}{1+3v}.$

(d) $\frac{4+x}{(1-x)(4+x^2)}=\frac{A}{1-x}+\frac{Bx+C}{4+x^2},$ where $A=\lim_{x\to 1}\frac{4+x}{4+x^2}=1,$

$B=1$ by equating the coefficients of x^2 and $C=0$ by equating the constants.

$$\int_2^n\frac{4+x}{(1-x)(4+x^2)}dx=\int_2^n\frac{-1}{x-1}dx+\int_2^n\frac{x}{4+x^2}dx$$
$$=\left[-\ln|x-1|+\frac{1}{2}\ln\left(4+x^2\right)\right]_2^n$$
$$=\left[-\frac{1}{2}\ln\left(x-1\right)^2+\frac{1}{2}\ln\left(4+x^2\right)\right]_2^n$$
$$=\frac{1}{2}\left[\ln\frac{4+x^2}{(x-1)^2}\right]_2^n$$
$$=\frac{1}{2}\left[\ln\frac{4+n^2}{(n-1)^2}-\ln 8\right]$$
$$=\frac{1}{2}\ln\frac{4+n^2}{8(n-1)^2}.$$

$\therefore f(n)=4+n^2.$

(e) $w=\frac{e^{i2\theta}-1}{e^{i2\theta}+1}=\frac{e^{i2\theta}-1}{e^{i2\theta}+1}\times\frac{e^{-i\theta}}{e^{-i\theta}}=\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}=\frac{2i\sin\theta}{2\cos\theta}=i\tan\theta,$

which is purely imaginary.

Question 13

(a) We'll prove by contrapositive, i.e. if n is even, where $n \ge 3$, then $2^n - 1$ is not a prime number.

Let $n = 2k$, where $k \ge 2$,

$$
\begin{aligned}
2^n - 1 &= 2^{2k} - 1 \\
&= \left(2^k\right)^2 - 1 \\
&= \left(2^k + 1\right)\left(2^k - 1\right).
\end{aligned}
$$

$\therefore 2^n - 1$ is not a prime number, as it is a product of 2 numbers, none of which is equal 1, for $k \ge 2$.

(b) Let $n = 1, a_1 = 2\cos\frac{\pi}{4} = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$. $\therefore$ True for $n = 1$.

Assume $a_n = 2\cos\frac{\pi}{2^{n+1}}$ for some value of n.

Required to prove that $a_{n+1} = 2\cos\frac{\pi}{2^{n+2}}$.

$$
\begin{aligned}
\text{LHS} = a_{n+1} &= \sqrt{2 + a_n} \\
&= \sqrt{2 + 2\cos\frac{\pi}{2^{n+1}}} \\
&= \sqrt{2}\sqrt{1 + \cos\frac{\pi}{2^{n+1}}} \\
&= \sqrt{2}\sqrt{2\cos^2\frac{\pi}{2 \times 2^{n+1}}}, \text{ as } 1 + \cos x = 2\cos^2\frac{x}{2} \\
&= 2\cos\frac{\pi}{2^{n+2}} = \text{RHS}.
\end{aligned}
$$

$\therefore$ It is true for all $n \ge 1$ by the principle of Induction.

(c) (i) $\sqrt[5]{-1} = \left(\text{cis}(\pi + 2k\pi)\right)^{\frac{1}{5}} = \text{cis}\frac{\pi + 2k\pi}{5}, k = 0, \pm1, \pm2.$

$\therefore z = \text{cis}\left(\pm\frac{\pi}{5}\right), \text{cis}\left(\pm\frac{3\pi}{5}\right), -1.$

(ii) $z^5 + 1 = (z+1)\left(z^4 - z^3 + z^2 - z + 1\right)$

If z is a solution of $z^5 + 1 = 0$ and $z \ne -1$ then z satisfies

$z^4 - z^3 + z^2 - z + 1 = 0.$

$z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0,$ on dividing by z^2, noting $z \ne 0$

$u^2 - u - 1 = 0,$ leting $u = z + \frac{1}{z}$ then $z^2 + \frac{1}{z^2} = u^2 - 2.$

(iii) Let $z = \text{cis}\frac{3\pi}{5}, z + \frac{1}{z} = 2\cos\frac{3\pi}{5}.$

Solving $u^2 - u - 1 = 0$ gives $u = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$

$\therefore 2\cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{2},$ since $\cos\frac{3\pi}{5} < 0$ (2nd quadrant).

$\therefore \cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}.$

Question 14

(a) (i) Assume $\lambda \ne 0$, dividing both sides of $\lambda\vec{u} + \mu\vec{v} = \vec{0}$ gives $\vec{u} = -\frac{\mu}{\lambda}\vec{v}$, i.e. $\vec{u}$ and $\vec{v}$ are parallel. The same argument applies by assuming $\mu \ne 0$.

By contradiction, $\lambda = \mu = 0$.

(ii) Arranging $\lambda_1\vec{u} + \mu_1\vec{v} = \lambda_2\vec{u} + \mu_2\vec{v}$ gives

$\left(\lambda_1 - \lambda_2\right)\vec{u} + \left(\mu_1 - \mu_2\right)\vec{v} = 0.$

$\therefore$ From part (i), $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.

(iii)
$$
\begin{aligned}
\overrightarrow{SK} &= k\overrightarrow{SL} \\
&= k\left(\overrightarrow{SB} + \overrightarrow{BL}\right) \\
&= k\left(\overrightarrow{SB} + \ell\overrightarrow{BC}\right) \\
&= k\overrightarrow{SB} + k\ell\left(\overrightarrow{SC} - \overrightarrow{SB}\right) \\
&= k(1-\ell)\overrightarrow{SB} + k\ell\overrightarrow{BC}.
\end{aligned}
$$

From data, $k(1-\ell) = \frac{1}{4}$ (1)

and $k\ell = \frac{1}{3}.$ (2)

$\frac{(1)}{(2)}$ gives $\frac{1-\ell}{\ell} = \frac{3}{4}$

$4 = 7\ell$

$\ell = \frac{4}{7}.$

$\therefore \overrightarrow{BL} = \frac{4}{7}\overrightarrow{BC}.$

(iv)
$$
\begin{aligned}
\overrightarrow{AP} &= -6\overrightarrow{AB} - 8\overrightarrow{AC} \\
&= -6\overrightarrow{AB} - 8\left(\overrightarrow{AB} + \overrightarrow{BC}\right) \\
&= -14\overrightarrow{AB} - 8\overrightarrow{BC} \\
&= -14\left(\overrightarrow{AB} + \frac{4}{7}\overrightarrow{BC}\right) \\
&= -14\left(\overrightarrow{AB} + \overrightarrow{BL}\right) \\
&= -14\overrightarrow{AL}.
\end{aligned}
$$

$\therefore$ Point P belongs to the line AL.

(b) (i)
$$
\begin{aligned}
J_0 &= \int_0^1 e^{-x}\,dx \\
&= \left[-e^{-x}\right]_0^1 \\
&= 1 - \frac{1}{e}.
\end{aligned}
$$

(ii) $J_n = \int_0^1 x^n e^{-x}\,dx \le \int_0^1 x^n\,dx$, since $0 < e^{-x} \le 1$ for $0 \le x \le 1$

$\therefore J_n \le \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}.$

(iii) Let $u = x^n, dv = e^{-x}dx$ then $du = nx^{n-1}, v = -e^{-x}$.

By Integration by parts, for $n \geq 1$,

$$J_n = \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}dx$$

$$= -\frac{1}{e} + nJ_{n-1}.$$

(iv) $J_n = nJ_{n-1} - \frac{1}{e}$

$$= n\left((n-1)J_{n-2} - \frac{1}{e}\right) - \frac{1}{e}$$

$$= n(n-1)J_{n-2} - \frac{1}{e}(1+n)$$

$$= n(n-1)J_{n-2} - \frac{n!}{e}\left(\frac{1}{n!} + \frac{1}{(n-1)!}\right)$$

$$= \ldots$$

$$= n(n-1)\ldots 1J_0 - \frac{n!}{e}\left(\frac{1}{n!} + \frac{1}{(n-1)!} + \ldots + \frac{1}{1!}\right)$$

$$= n!J_0 - \frac{n!}{e}\sum_{r=1}^{n}\frac{1}{r!}$$

$$= n!\left(1 - \frac{1}{e}\right) - \frac{n!}{e}\sum_{r=1}^{n}\frac{1}{r!}$$

$$= n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!} \text{ for all } n \geq 0.$$

Note: I have chosen not to prove by Induction.

(v) $n! - \frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!} \leq \frac{1}{n+1}$

$$1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!} \leq \frac{1}{n!(n+1)}$$

$$1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!} \leq \frac{1}{(n+1)!}.$$

As $J_n \geq 0, \forall n$, when $n \to \infty, 0 \leq 1 - \frac{1}{e}\sum_{r=0}^{\infty}\frac{1}{r!} \leq 0$.

$\therefore 1 - \frac{1}{e}\sum_{r=0}^{\infty}\frac{1}{r!} = 0$, by the sandwich principle.

$$\therefore \frac{1}{e}\sum_{r=0}^{\infty}\frac{1}{r!} = 1$$

$$\therefore e = \sum_{r=0}^{\infty}\frac{1}{r!}.$$

Question 15

(a) (i) Resolving the forces

horizontally, $T_1\cos\theta = T_2\cos\phi$ (1)

vertically, $T_1\sin\theta - T_2\sin\phi - Mg = Ma = 0$, as the machine moves upwards with constant velocity.

$\therefore T_1\sin\theta = T_2\sin\phi + Mg$ (2)

$\frac{(2)}{(1)}$ gives $\tan\theta = \tan\phi + \frac{Mg}{T_2\cos\phi}$. (3)

(ii) From (3), $\tan\theta > \tan\phi$, since $\frac{Mg}{T_2\cos\phi} > 0$.

$$\therefore \frac{\ell}{d} > \frac{h-\ell}{2d}$$

$$2\ell > h - \ell$$

$$\ell > \frac{h}{3}$$

$h - \ell < \frac{2h}{3}$, $\therefore$ the machine cannot be lifted higher than $\frac{2h}{3}$.

(b) Period $= \frac{1}{\text{Frequency}} = \frac{1}{40} = \frac{2\pi}{n}, \therefore n = 80\pi$.

Centre of motion $= \frac{0.17+0.05}{2} = 0.11$ m.

$\ddot{y} = -n^2(y - 0.11)$

Max. acceleration occurs at $y = 0.05$ and 0.17 m.

$\ddot{y}_{\max} = -n^2(0.05 - 0.11) = 0.06n^2 = 0.06\times(80\pi)^2$.

$\therefore$ The maximum force $= m\ddot{y}_{\max} = 0.8\times 0.06\times(80\pi)^2$

$\simeq 3031$ newtons.

(c) $x = \tan^2\theta, dx = 2\tan\theta\sec^2\theta d\theta$.

When $x = 0, \theta = 0$; when $x = 1, \theta = \frac{\pi}{4}$.

$$I = \int_0^1 \sin^{-1}\sqrt{\frac{x}{x+1}}dx = \int_0^{\frac{\pi}{4}} \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}2\tan\theta\sec^2\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta\sec^2\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta\tan\theta\sec^2\theta d\theta.$$

Let $u = \theta, dv = \tan\theta\sec^2\theta d\theta$ then $du = d\theta, v = \frac{1}{2}\tan^2\theta$

$$I = \left[\theta\tan^2\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\tan^2\theta d\theta$$

$$= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}}\left(\sec^2\theta - 1\right)d\theta$$

$$= \frac{\pi}{4} - \left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} - 1.$$

(d) $|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right| \le \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right|$

$|z| \le 2 + \frac{4}{|z|}$, since $\left|z - \frac{4}{z}\right| = 2$, by data.

$\therefore |z|^2 - 2|z| - 4 \le 0.$

$(|z| - 1)^2 \le 5$

$-\sqrt{5} \le |z| - 1 \le \sqrt{5}$

$1 - \sqrt{5} \le |z| \le 1 + \sqrt{5}.$

Question 16

(a) Let $A = z_A = 5 + i, B = z_B = \alpha + 5i, C = z_C = \beta - 5i.$

$\overrightarrow{AB}$ rotated $\frac{\pi}{3} = \overrightarrow{AC}$

$(z_B - z_A)\text{cis}\frac{\pi}{3} = z_C - z_A$

$(\alpha - 5 + 4i)\left(\frac{1 + i\sqrt{3}}{2}\right) = \beta - 5 - 6i$

Equating the imaginary parts gives

$\frac{\sqrt{3}}{2}(\alpha - 5) + 2 = -6.$

$\alpha - 5 = -\frac{16}{\sqrt{3}}$

$\alpha = 5 - \frac{16}{\sqrt{3}}.$

$\therefore z_B = 5 - \frac{16}{\sqrt{3}} + 5i.$

(b) $M\ddot{y} = -Mg - 0.1Mv$

$\therefore \ddot{y} = -(10 + 0.1v)$, taking $g = 10$

$\ddot{y} = \frac{dv}{dt} = -\frac{100 + v}{10}.$

$\int_{v_0}^{v} \frac{dv}{100 + v} = -\frac{1}{10}\int_0^{t_1} dt$

$\left[\ln(100 + v)\right]_{v_0}^{v} = -0.1t_1$

$\ln\frac{100 + v}{100 + v_0} = -0.1t.$

$\frac{100 + v}{100 + v_0} = e^{-0.1t}.$

$\therefore v = \frac{dy}{dt} = (100 + v_0)e^{-0.1t} - 100.$

$\int_0^0 dy = \int_0^7 \left[(100 + v_0)e^{-0.1t} - 100\right]dt$

$0 = -10(100 + v_0)\left[e^{-0.1t}\right]_0^7 - 700$

$0 = -10(100 + v_0)\left[e^{-0.7} - 1\right] - 700$

$(100 + v_0)\left[e^{-0.7} - 1\right] = -70$

$100 + v_0 = \frac{70}{1 - e^{-0.7}} = 139.05$

$\therefore v_0 = 39.05 = 39.1$ to 1 dp.

(c) (i) For the rectangular prism of sides a, b, c, its volume $V = abc$, and its surface area $S = 2ab + 2ac + 2bc$.

From given data, with $n = 3$,

$\frac{x_1x_2x_3}{A^3} \le 1$, where $A = \frac{x_1 + x_2 + x_3}{3}$

$\therefore x_1x_2x_3 \le \left(\frac{x_1+x_2+x_3}{3}\right)^3$. (1)

Let $x_1 = 2ab, x_2 = 2ac, x_3 = 2bc$

(1) becomes $8(abc)^2 \le \frac{S^3}{27}$

$(abc)^2 \le \frac{S^3}{216} = \frac{S^3}{6^3}$.

$\therefore abc \le \left(\frac{S}{6}\right)^{\frac{3}{2}}$.

(ii) From the result above, $V \le \left(\frac{S}{6}\right)^{\frac{3}{2}}$. $\therefore V$ is maximum when it is equal to $\left(\frac{S}{6}\right)^{\frac{3}{2}}$.

If the rectangular prism is a cube of its sides $= a$, then $V = a^3$ and $S = 6a^2$.

$\text{RHS} = \left(\frac{S}{6}\right)^{\frac{3}{2}} = \left(\frac{6a^2}{6}\right)^{\frac{3}{2}} = a^3 = \text{LHS}$.

$\therefore$ The cube has the maximum volume.

(d) $|z_1| = |z_2| = |z_3|$ geometrically means these complex numbers belong to the same circle of centre the origin.

Let $|z_1| = |z_2| = |z_3| = r$.

Taking the moduli both sides of $z_1z_2z_3 = 1$ gives

$r^3 = 1$, since $|z_1z_2z_3| = |z_1||z_2||z_3|$.

$\therefore |z_1| = |z_2| = |z_3| = 1$.

$\therefore z_1, z_2$ and z_3 are the roots of the equation with

Sum of the roots $\sum \alpha = z_1 + z_2 + z_3 = 1$ (given)

Product of the roots $\prod \alpha = z_1z_2z_3 = 1$ (given)

Sum of the product of 2 roots at a time $\sum \alpha\beta = z_1z_2 + z_1z_3$

$+z_2z_3 = z_1z_2z_3\left(\frac{1}{z_3} + \frac{1}{z_2} + \frac{1}{z_1}\right) = z_1z_2z_3 \times \left(\overline{z_1} + \overline{z_2} + \overline{z_3}\right)$

$= 1 \times \left(\overline{z_1 + z_2 + z_3}\right) = 1 \times 1 = 1$.

z_1, z_2 and z_3 satisfy the equation $z^3 - z^2 + z - 1 = 0$.

$z^2(z-1) + z - 1 = 0$

$(z^2+1)(z-1) = 0$.

$\therefore z = \pm i$ and 1.