Multiple Choice

1 A) $(3-2i)^2 = 9-4-2\times 3\times 2i = 5-12i$

2 A) $\displaystyle\int \frac{-du}{u^3} = \frac{1}{2u^2}, \therefore \frac{1}{2\cos^2 x} = \frac{1}{2}\sec^2 x.$

3 C) $u = x, dv = \cos x dx, du = dx, v = \sin x$

$$\therefore \int x\cos x dx = x\sin x - \int \sin x dx$$

$$= x\sin x + \cos x + C.$$

4 D) If $f(x) = 2(x-1)^2(x+3)$ then $b = 2$

If $f(x) = 2(x+3)^2(x-1)$ then $b = 10$

5 B) by definition, $PA + PB =$ constant, then the locus of P is an ellipse with foci A and B.

6 B) $y = \pm\sqrt{2\sin|x|}$, where $\sin|x| \geq 0$.

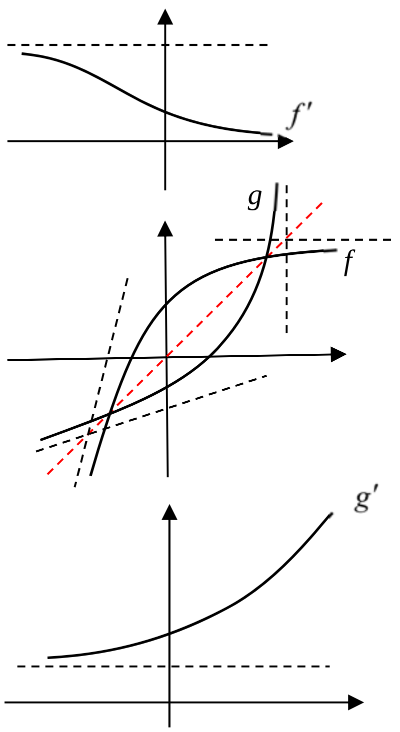
7 D) For $0 \leq x \leq \frac{\pi}{4}$, $\tan x > \tan^2 x$ and $1-\tan x < 1-\tan^2 x$

$\therefore$ A > B and D > C.

$\text{A} = \ln\sqrt{2} \simeq 0.35,\ \text{D} = \frac{\pi}{4} - \left(1-\frac{\pi}{4}\right) = \frac{\pi}{2} - 1 \simeq 0.57.$

8 C) $z = \text{cis}\frac{-\pi}{6}, -i\overline{z} = \text{cis}\left(\frac{-\pi}{2}+\frac{\pi}{6}\right) = \text{cis}\frac{-\pi}{3} = z^2$

9 D)



10 A) Configuration = (3,1) and (2,2)

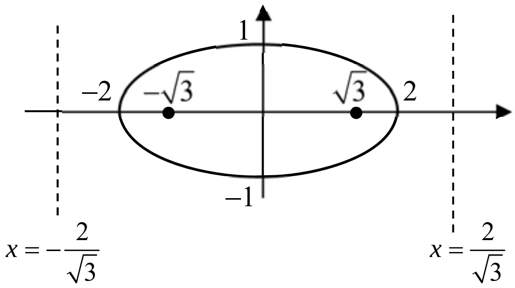
$$\therefore {}^{10}C_2\left(2!\times\frac{4!}{3!}+\frac{4!}{2!2!}\right) = 630$$

Question 11

(a) (i) $z + \overline{w} = (1+3i) + (2+i) = 3+4i$

(ii) $\dfrac{z}{w} = \dfrac{1+3i}{2-i} = \dfrac{(1+3i)(2+i)}{5} = \dfrac{-1+7i}{5}$

(b)



Foci $S\left(\pm\sqrt{4-1}, 0\right) = \left(\pm\sqrt{3}, 0\right)$

Directrices $x = \pm\dfrac{2}{\sqrt{3}}$

(c) $\displaystyle\int \frac{dx}{x^2+10x+29} = \int \frac{dx}{(x+5)^2+4} = \frac{1}{2}\tan^{-1}\frac{x+5}{2} + C$

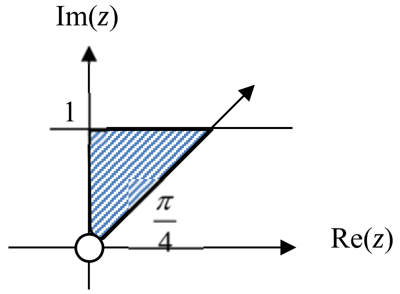
(d) $\displaystyle\int \frac{6dx}{(x+3)(x-3)} = \int\left(\frac{-1}{x+3}+\frac{1}{x-3}\right)dx = \ln\left|\frac{x-3}{x+3}\right| + C$

(e) (i) $z = -1 + i\sqrt{3} = 2\,\text{cis}\dfrac{2\pi}{3}$

(ii) $z^3 = 8\,\text{cis}\,2\pi = 8 = 8 + 0i$

Question 12

(a)



(b) (i) Using the Cosine rule,

$$x^2+y^2-2xy\cos\frac{2\pi}{3}=70^2$$

$$x^2+y^2+xy=70^2$$

(ii) $2x\frac{dx}{dt}+2y\frac{dy}{dt}+y\frac{dx}{dt}+x\frac{dy}{dx}=0$

$$2x\frac{dx}{dt}+8y+y\frac{dx}{dt}+4x=0$$

$$60\frac{dx}{dt}+400+50\frac{dx}{dt}+120=0$$

$$\frac{dx}{dt}=-\frac{520}{110}=-\frac{52}{11}\text{ ms}^{-1}.$$

(c) Volume $V=2\pi\int_{10}^{40}xydx=2\pi A\int_{10}^{40}xe^{-kx}dx$

Let $u=x, dv=e^{-kx}dx, du=dx, v=-\frac{1}{k}e^{-kx}$.

$$V=\left[-\frac{2\pi A}{k}xe^{-kx}\right]_{10}^{40}+\frac{2\pi A}{k}\int_{10}^{40}e^{-kx}dx$$

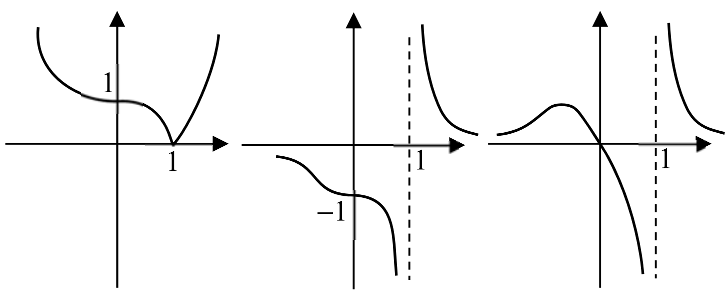
$$=\frac{2\pi A}{k}\left(10e^{-10k}-40e^{-40k}\right)-\frac{2\pi A}{k}\left[\frac{1}{k}e^{-kx}\right]_{10}^{40}$$

$$=\frac{2\pi A}{k}\left(10e^{-10k}-40e^{-40k}\right)+\frac{2\pi A}{k^2}\left(e^{-10k}-e^{-40k}\right)$$

$$=\frac{2\pi A}{k^2}\left((10k+1)e^{-10k}-(40k+1)e^{-40k}\right)$$

(d)

(i) (ii) (iii)



Question 13

(a) (i) $m=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b\sec\theta}{a\tan\theta}$.

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$

$$ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$$

$$bx\sec\theta-ay\tan\theta=ab\left(\sec^2\theta-\tan^2\theta\right)=ab.$$

(ii) $bx\sec\theta-ay\tan\theta=ab$ (1)

$bx\sec\phi-ay\tan\phi=ab$ (2)

$(2)\times\tan\theta-(1)\times\tan\phi$ gives

$$bx\left(\sec\phi\tan\theta-\sec\theta\tan\phi\right)=ab\left(\tan\theta-\tan\phi\right)$$

$$x=\frac{a\left(\tan\theta-\tan\phi\right)}{\sec\phi\tan\theta-\sec\theta\tan\phi}.$$

(iii) $m_{OT}=\dfrac{\dfrac{b\left(\sec\theta-\sec\phi\right)}{\sec\phi\tan\theta-\sec\theta\tan\phi}}{\dfrac{a\left(\tan\theta-\tan\phi\right)}{\sec\phi\tan\theta-\sec\theta\tan\phi}}=\dfrac{b\left(\sec\theta-\sec\phi\right)}{a\left(\tan\theta-\tan\phi\right)}$

Eqn of OT : $y=\frac{b\left(\sec\theta-\sec\phi\right)}{a\left(\tan\theta-\tan\phi\right)}x.$ (3)

Sub $M\left(\frac{a}{2}(\sec\theta+\sec\phi),\frac{b}{2}(\tan\theta+\tan\phi)\right)$ to (3),

LHS $=\frac{b}{2}(\tan\theta+\tan\phi)$.

RHS $=\frac{b\left(\sec\theta-\sec\phi\right)}{a\left(\tan\theta-\tan\phi\right)}\times\frac{a}{2}(\sec\theta+\sec\phi)$

$$=\frac{b}{2}\frac{\sec^2\theta-\sec^2\phi}{\tan\theta-\tan\phi}$$

$$=\frac{b}{2}\frac{\tan^2\theta+1-\tan^2\phi-1}{\tan\theta-\tan\phi}$$

$$=\frac{b}{2}\frac{\tan^2\theta-\tan^2\phi}{\tan\theta-\tan\phi}$$

$$=\frac{b}{2}(\tan\theta+\tan\phi)=\text{LHS}.$$

$\therefore O,T$ and M are collinear.

(b) (i) Area$(ABCD)=\frac{1}{2}(AD+BC)AB$

$$=\frac{1}{2}(1+(1-y))y=\frac{1}{2}\left(2-(1-x^2)\right)(1-x^2)$$

$$=\frac{1}{2}\left(1+x^2\right)\left(1-x^2\right)=\frac{1}{2}\left(1-x^4\right)$$

(ii) $V=2\times\frac{1}{2}\int_0^1\left(1-x^4\right)dx=\left[x-\frac{x^5}{5}\right]_0^1=\frac{4}{5}\text{ u}^3.$

(c) (i) $y=0, t=\frac{v\sin\theta}{4.9}$,

$$x=\frac{v^2\sin\theta\cos\theta}{4.9}=\frac{20^2\sin\frac{\pi}{3}\cos\frac{\pi}{3}}{4.9}=\frac{100\sqrt{3}}{4.9}\text{ m.}$$

(ii) For object 2, when $t=\dfrac{20\sin\frac{\pi}{3}}{4.9}-2=\dfrac{10\sqrt{3}}{4.9}-2$, $x=\dfrac{100\sqrt{3}}{4.9}$, $y=0$. Let u be its initial speed and θ be its angle of projection.

$$\frac{100\sqrt{3}}{4.9}=u\left(\frac{10\sqrt{3}}{4.9}-2\right)\cos\theta$$

$$100\sqrt{3}=u\left(10\sqrt{3}-9.8\right)\cos\theta \qquad (1)$$

$$0=-4.9\left(\frac{10\sqrt{3}}{4.9}-2\right)^2+u\left(\frac{10\sqrt{3}}{4.9}-2\right)\sin\theta$$

$$4.9\left(\frac{10\sqrt{3}}{4.9}-2\right)=u\sin\theta$$

$$10\sqrt{3}-9.8=u\sin\theta \qquad (2)$$

$\dfrac{(1)}{(2)}$ gives $\dfrac{100\sqrt{3}}{10\sqrt{3}-9.8}=\left(10\sqrt{3}-9.8\right)\cot\theta.$

$$\therefore \tan\theta=\frac{\left(10\sqrt{3}-9.8\right)^2}{100\sqrt{3}}\simeq 0.3265$$

$\therefore \theta\simeq 0.32$ radians or $18°$.

Sub. to (2), $u\simeq\dfrac{10\sqrt{3}-9.8}{\sin 18°}=24.3\text{ ms}^{-1}$.

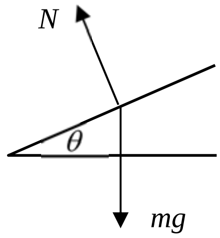
Question 14

(a) Resolving the forces

vertically, $N\cos\theta=mg$ (1)

horizontally, $N\sin\theta=\dfrac{mv^2}{r}$ (2)

$\dfrac{(2)}{(1)}$ gives $\tan\theta=\dfrac{v^2}{rg}$, $\therefore v^2=gr\tan\theta$.



(b) (i) $\ddot{x}=0$ gives $g-kw=0$, $\therefore w=\dfrac{g}{k}$.

(ii) $\displaystyle\int_{1.6w}^{1.1w}\frac{dv}{g-kv}=\int_0^T dt$, letting $\ddot{x}=\dfrac{dv}{dt}$

$$T=\left[-\frac{1}{k}\ln|g-kv|\right]_{1.6w}^{1.1w}$$

$$=\frac{1}{k}\ln\left|\frac{g-1.6wk}{g-1.1wk}\right|$$

$$=\frac{1}{k}\ln\left|\frac{g-1.6g}{g-1.1g}\right|$$

$$=\frac{1}{k}\ln\frac{0.6}{0.1}$$

$$=\frac{1}{k}\ln 6\text{ s.}$$

(iii) $\displaystyle\int_{1.6w}^{1.1w}\frac{vdv}{g-kv}=\int_0^D dx$, letting $\ddot{x}=\dfrac{vdv}{dx}$

$$D=-\frac{1}{k}\int_{1.6w}^{1.1w}\left(1-\frac{g}{g-kv}\right)dv$$

$$=-\frac{1}{k}\left[v+\frac{g}{k}\ln|g-kv|\right]_{1.6w}^{1.1w}$$

$$=\frac{1}{k}(1.6w-1.1w)+\frac{g}{k^2}\ln\left|\frac{g-1.6g}{g-1.1g}\right|$$

$$=\frac{g}{2k^2}+\frac{g}{k^2}\ln 6$$

$$=\frac{g}{k^2}\left(\frac{1}{2}+\ln 6\right)$$

(c) (i) Let $t = \tan x$

$$\cot x - \cot 2x = \frac{1}{t} - \frac{1-t^2}{2t}$$
$$= \frac{1+t^2}{2t}$$
$$= \operatorname{cosec} 2x.$$

(ii) Let $n = 1$, $\text{LHS} = \operatorname{cosec} 2x = \cot x - \cot 2x$

$= \text{RHS}$, from part (i).

Assume $\exists n : \sum_{r=1}^{n} \operatorname{cosec}\left(2^r x\right) = \cot x - \cot\left(2^n x\right)$.

RTP $\sum_{r=1}^{n+1} \operatorname{cosec}\left(2^r x\right) = \cot x - \cot\left(2^{n+1} x\right)$

$$\text{LHS} = \cot x - \cot\left(2^n x\right) + \operatorname{cosec}\left(2^{n+1} x\right)$$
$$= \cot x - \cot\left(2^n x\right) + \operatorname{cosec}\left(2 \times 2^n x\right)$$
$$= \cot x - \cot\left(2^n x\right) + \cot\left(2^n x\right) - \cot\left(2 \times 2^n x\right)$$
$$= \cot x - \cot\left(2^{n+1} x\right) = \text{RHS}.$$

$\therefore$ True for all $n \geq 1$ by the principle of Induction.

Question 15

(a) (i) Let $u = -x, du = -dx$.

$$\int_{-a}^{a} \frac{f(x)}{f(x)+f(-x)}dx = \int_{a}^{-a} \frac{f(-u)}{f(-u)+f(u)}(-du)$$
$$= \int_{-a}^{a} \frac{f(u)}{f(u)+f(-u)}du$$
$$= \int_{-a}^{a} \frac{f(x)}{f(x)+f(-x)}dx.$$

(ii) $\int_{-1}^{1} \frac{e^x}{e^x+e^{-x}}dx = \int_{-1}^{1} \frac{e^{-x}}{e^x+e^{-x}}dx.$

$$\therefore \int_{-1}^{1} \frac{e^x}{e^x+e^{-x}}dx = \frac{1}{2}\int_{-1}^{1} \frac{e^x+e^{-x}}{e^x+e^{-x}}dx$$
$$= \frac{1}{2}\int_{-1}^{1} dx$$
$$= \frac{1}{2}\Big[x\Big]_{-1}^{1}$$
$$= 1.$$

(b) (i)

$$\Pr(A) = \underbrace{\frac{w}{w+y}}_{\text{A wins}} + \underbrace{\frac{y}{w+y}}_{\text{A misses}} \times \underbrace{\frac{w}{w+y}}_{\text{B misses}} \times \underbrace{\frac{w}{w+y}}_{\text{A wins}}$$
$$+\left(\frac{y}{w+y}\frac{w}{w+y}\right)^2 \frac{w}{w+y} + \ldots$$

$$= \frac{\frac{w}{w+y}}{1-\frac{wy}{(w+y)^2}}, \text{ using limiting sum of a GP}$$
$$= \frac{w^2+wy}{w^2+y^2+wy}.$$

If $w = y, \Pr(A) = \frac{2w^2}{3w^2} = \frac{2}{3} > \frac{1}{2}, \therefore A$ has a higher chance to win.

(ii) $\Pr(B) = 1 - \Pr(A)$

$$= 1 - \frac{w^2+wy}{w^2+y^2+wy}$$
$$= \frac{y^2}{w^2+y^2+wy}.$$

If $\Pr(B) > \Pr(A), y^2 > w^2 + wy$

$$\frac{y^2}{w^2} - \frac{y}{w} - 1 > 0.$$

$$\therefore \frac{y}{w} > \frac{1+\sqrt{5}}{2} \quad \left(\text{reject } \frac{y}{w} < \frac{1-\sqrt{5}}{2} < 0\right)$$

(c) (i) $\int_0^1 \frac{x}{(x+1)^2}dx = \int_0^1 \frac{x+1-1}{(x+1)^2}dx$

$= \int_0^1 \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx$

$= \left[\ln(x+1) + \frac{1}{x+1}\right]_0^1$

$= \ln 2 + \frac{1}{2} - 1$

$= \ln 2 - \frac{1}{2}.$

(ii) Let $u = x^n, dv = \frac{dx}{(x+1)^2}, du = nx^{n-1}dx, v = \frac{-1}{x+1}$

$I_n = \int_0^1 \frac{x^n}{(x+1)^2}dx = \left[\frac{-x^n}{x+1}\right]_0^1 + n\int_0^1 \frac{x^{n-1}}{x+1}dx$

$= -\frac{1}{2} + n\int_0^1 \frac{x^{n-1}(x+1)}{(x+1)^2}dx$

$= -\frac{1}{2} + n\left(I_n + I_{n-1}\right)$

$\therefore (1-n)I_n = -\frac{1}{2} + nI_{n-1}.$

$\therefore I_n = \frac{1}{2(n-1)} - \frac{n}{n-1}I_{n-1}.$

(iii) $I_3 = \frac{1}{4} - \frac{3}{2}I_2$

$I_2 = \frac{1}{2} - 2I_1$

$I_1 = \ln 2 - \frac{1}{2}$, from part (i).

$\therefore I_3 = \frac{1}{4} - \frac{3}{2}\left(\frac{1}{2} - 2\ln 2 + 1\right)$

$= \frac{1}{4} - \frac{3}{4} + 3\ln 2 - \frac{3}{2}$

$= 3\ln 2 - 2$

Question 16

(a) Substituting $x = r\cos\theta$,

$r^3\cos^3\theta - pr\cos\theta + q = r\left(r^2\cos^3\theta - p\cos\theta\right) + q$

$= r\left(\frac{4p}{3}\cos^3\theta - p\cos\theta\right) + q$

$= \frac{rp}{3}\left(4\cos^3\theta - 3\cos\theta\right) + q$

$= \frac{rp}{3}\cos 3\theta + q$

$= \frac{rp}{3} \times \frac{-4q}{r^3} + q$

$= -\frac{4pq}{3r^2} + q$

$= -q + q$, since $r^2 = \frac{4p}{3}$,

$= 0.$

$\therefore r\cos\theta$ is a root of $x^3 - px + q = 0.$

(ii) Let $x = \alpha + 3, \therefore \alpha = x - 3.$

Substituting x by $x - 3$,

$(x-3)^3 + 9(x-3)^2 + 15(x-3) - 17 = 0$

$x^3 - 9x^2 + 27x - 27 + 9x^2 - 54x + 81 + 15x - 45 - 17 = 0.$

$x^3 - 12x - 8 = 0.$

(iii) The roots of $x^3 - 12x - 8 = 0$ are $x = r\cos\theta$, where

$r = \sqrt{\frac{4\times 12}{3}} = 4$, and $\cos 3\theta = \frac{-4\times -8}{64} = \frac{1}{2}$,

$\therefore 3\theta = \pm\frac{\pi}{3} + 2k\pi.$

$\therefore \theta = \pm\frac{\pi}{9} + \frac{2k\pi}{3}$

$= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$, taking $k = 0, 1.$

$\therefore x = 4\cos\frac{\pi}{9}, 4\cos\frac{5\pi}{9}, 4\cos\frac{7\pi}{9}.$

$\therefore$ The roots of $x^3 + 9x^2 + 15x - 17 = 0$ are $x = 4\cos\frac{\pi}{9} - 3$,

$4\cos\frac{5\pi}{9} - 3, 4\cos\frac{7\pi}{9} - 3.$

(b) (i) $\sum\alpha = 2\left(\text{Re}(\alpha) + \text{Re}(\beta)\right) = 2k,$

$\therefore \text{Re}(\alpha) + \text{Re}(\beta) = k.$ (1)

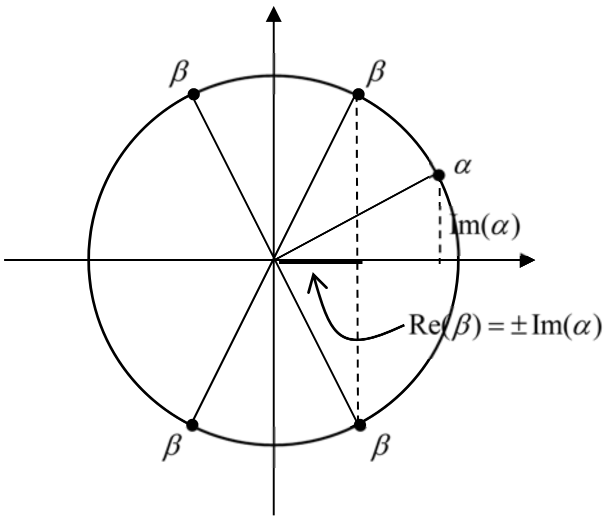
$\sum\alpha\beta = 2 + 4\left(\text{Re}(\alpha)\text{Re}(\beta)\right) = 2k^2,$

$\therefore 1 + 2\,\text{Re}(\alpha)\text{Re}(\beta) = k^2.$ (2)

$(1)^2 - (2)$ gives $\left(\text{Re}(\alpha)\right)^2 + \left(\text{Re}(\beta)\right)^2 = 1$

(ii) But $\left(\text{Re}(\alpha)\right)^2 + \left(\text{Im}(\alpha)\right)^2 = 1,$

$\therefore \text{Re}(\beta) = \pm\text{Im}(\alpha).$



(c) By the Sine rule,

$$\frac{d}{\sin BCA} = \frac{e}{\sin BAC}$$

$$\text{LHS} = \frac{d}{\sin(BCD - ACD)}$$

$$= \frac{d}{\sin(BCD - (\pi - E))}$$ (opposite angles in a cyclic quad are supplementary)

$$= \frac{d}{\sin(C + E - \pi)}$$

$$= -\frac{d}{\sin(C + E)}.$$

$$\text{RHS} = \frac{e}{\sin BAC}$$

$$= \frac{e}{\sin BDC},$$ since $\angle BAC = \angle BDC$ (angles subtending the same arc are equal)

$$= \frac{a}{\sin CBD}$$

$$= \frac{a}{\sin(CBA - DBA)}$$

$$= \frac{a}{\sin(B - (\pi - E))},$$ (opposite angles in a cyclic quad are supplementary)

$$= -\frac{a}{\sin(B + E)}$$

$$\therefore \frac{a}{\sin(B + E)} = \frac{d}{\sin(C + E)}.$$