Multiple Choice

1 B) $\frac{1}{2}\sin^{-1}2x + C$

2 C) $\frac{x^2}{4} - \frac{y^2}{9} = 1, \therefore$ asymptotes $y = \pm\frac{3x}{2}$

3 D) $\alpha = -\frac{1}{x}, \therefore -\frac{1}{x^3} + \frac{2}{x^2} - \frac{5}{x} - 1 = 0,$

$\therefore x^3 + 5x^2 - 2x + 1 = 0$

4 C) $y = \frac{1}{\sqrt{x^2 - 4}}$

5 A) Area $= \pi(y+1)^2,$

$\therefore V = \pi\int_0^4 (y+1)^2\,dx = \pi\int_0^4 (e^{3x}+1)^2\,dx$

6 A) $i = \text{cis}\left(\frac{\pi}{2} + 2k\pi\right), \therefore i^{\frac{1}{6}} = \text{cis}\left(\frac{\pi}{12} + \frac{k\pi}{3}\right).$

(A) is when $k = 2$

7 D) $\arg(z) = \arg(z+1-i) \Leftrightarrow \arg\frac{z+1-i}{z-0} = 0$

8 B) The concavity of $F(x)$ changes when its 2nd derivative, which is $f'(x)$, changes signs.

9 B) Let $p = ix, q = iy$, where x and y are real,

$(ap-bq)^2 = (iax - iby)^2 = -(ax-by)^2 \le 0$

$(ab - pq) = (ab+mn)^2 \ge 0$

10 C) $f'(x) = \cos x = 0$ when $x = \frac{\pi}{2} + k\pi$

$g'(x) = \sin x + x\cos x = 0$ when $\tan x = -x$

$\therefore$ The line $y = -x$ meets the asymptotes $x = \frac{\pi}{2} + k\pi$

at b, where $|b| > |a|$

Question 11

(a) (i) $zw = (2+3i)(1-i) = 5+i$

(ii) $\overline{z} - \frac{2}{w} = 2 - 3i - \frac{2}{1-i} = 2 - 3i - (1+i) = 1 - 4i$

(b) $p'(4) = 0 \Rightarrow 48 + 8a = 0, \therefore a = -6$

$p(4) = 0 \Rightarrow 64 + 16a + b = 0, \therefore b = 32$

$p(r) = 0 \Rightarrow r^3 - 6r^2 + 32 = 0, \therefore r = -2$

(c) $\frac{x^2 - x - 6}{(x+1)(x^2-3)} = \frac{2}{x+1} + \frac{-x}{x^2-3}$

$\int\left(\frac{2}{x+1} + \frac{-x}{x^2-3}\right)dx = 2\ln(x+1) - \frac{1}{2}\ln(x^2-3) + C$

(d) (i) $w = iu = -2 + 5i$

(ii) $v = u + w = 3 + 7i$

(iii) $\arg\frac{w}{v} = \frac{\pi}{4}$

(e) $d^2 = r^2 + r^2 - 2r^2\cos\angle AOC = 2r^2\left(1 - \cos(2D)\right)$

$= 2r^2 \times 2\sin^2 D, \therefore d = 2r\sin D$

Question 12

(a) Area $= \frac{1}{2}x^2 \times \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}x^2$

Volume $= \frac{\sqrt{3}}{4}\int_{-1}^{1} x^2 dy = \frac{\sqrt{3}}{2}\int_0^1 \left(1-y^2\right)^2 dy$

$= \frac{\sqrt{3}}{2}\int_0^1 \left(1-2y^2+y^4\right)dy$

$= \frac{\sqrt{3}}{2}\left[y - \frac{2y^3}{3} + \frac{y^5}{5}\right]_0^1 = \frac{4\sqrt{3}}{15}$ units^3

(b) (i) $2x + y + xy' + 2yy' = 0$

$y'(x+2y) = -(2x+y)$

$y' = -\frac{2x+y}{x+2y}$

(ii) $y' = 0$ when $y = -2x$.

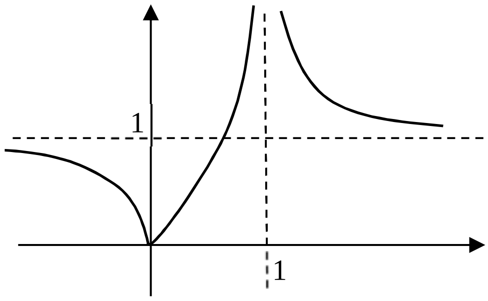
Sub $y = -2x$ gives $x^2 - 2x^2 + 4x^2 = 3, \therefore 3x^2 = 3, \therefore x = \pm 1$

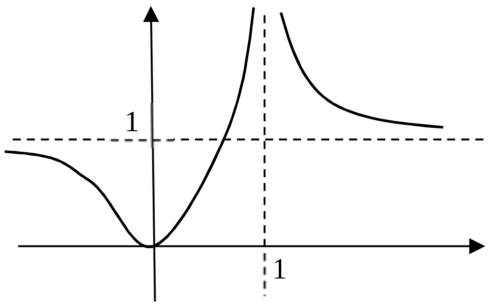
The points are $(1,-2)$ and $(-1,2)$

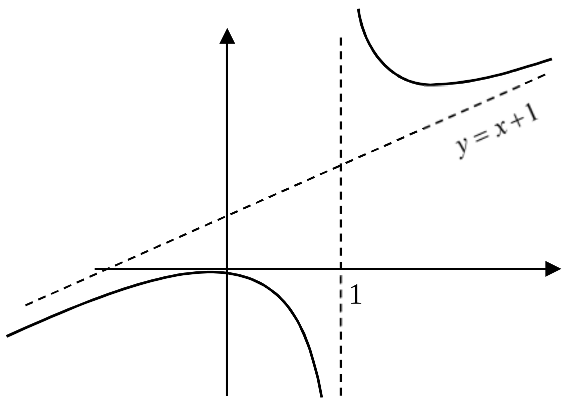
(c) $\int \frac{x^2+2x}{x^2+2x+5}dx = \int\left(1 - \frac{5}{(x+1)^2+4}\right)dx$

$= x - \frac{5}{2}\tan^{-1}\frac{x+1}{2} + C$

(d)







Question 13

(a) $V = 2\pi\int_0^1 (1-x)(2y)dx = 4\pi\int_0^1 (1-x)^2\sqrt{x}dx$

$= 4\pi\int_0^1 \left(\sqrt{x} - 2x\sqrt{x} + x^2\sqrt{x}\right)dx$

$= 4\pi\left[\frac{2x\sqrt{x}}{3} - \frac{4x^2\sqrt{x}}{5} + \frac{2x^3\sqrt{x}}{7}\right]_0^1$

$= \frac{64\pi}{105}$ units^2

(b) (i) $z = 1 - \cos 2\theta + i\sin 2\theta = 2\sin^2\theta + i2\sin\theta\cos\theta$

$= 2\sin\theta\left(\sin\theta + i\cos\theta\right)$

$= i2\sin\theta\left(\cos\theta - i\sin\theta\right)$

$\therefore |z| = 2\sin\theta$

(ii) $\arg(z) = \arg(i) + \arg\left(\cos\theta - i\sin\theta\right)$

$= \arg(i) + \arg\frac{1}{\cos\theta + i\sin\theta}$

$= \arg(i) - \arg\left(\cos\theta + i\sin\theta\right) = \frac{\pi}{2} - \theta$

(c) Resolving the forces

vertically, $N + T\cos\theta = mg$

horizontally, $T\sin\theta = m\ell\sin\theta\omega^2, \therefore T = m\ell\omega^2$

$N + m\ell\cos\theta\omega^2 = mg$

$N \ge 0, \therefore mg \ge m\ell\cos\theta\omega^2, \therefore \omega^2 \le \frac{g}{\ell\cos\theta}$

(d) (i) $x + pqy = c(p+q)$

$x + \frac{c^2pq}{x} = c(p+q)$

$x^2 - c(p+q)x + c^2pq = 0$

Midpoint of PQ has x-coordinate $x = \frac{\sum\alpha}{2} = \frac{c(p+q)}{2}$

But $x_S = 0, x_R = c(p+q), \therefore$ the midpoint of RS has x-coordinate $x = \frac{c(p+q)}{2}$, too.

As P, Q, R, S are collinear and PQ and RS have the same midpoint, $PS = RQ$.

(ii) Re-arranging $y = tx - at^2$ as $x - \frac{1}{t}y = at$

$\therefore$ from (i), $x_M = \frac{\sum\alpha}{2} = \frac{at}{2}$ (1)

$y_M = \frac{at^2}{2} - at^2 = -\frac{at^2}{2}$ (2)

From (1), $t = \frac{2x}{a}$

$\therefore y = -\frac{a}{2}\times\frac{4x^2}{a^2} = -\frac{2x}{a}$

$\therefore 2x^2 = -ay$

Question 14

(a) Let $t=\tan\frac{\theta}{2}, dt=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta, \therefore d\theta=\frac{2dt}{1+t^2}$

When $\theta=0, t=0$; when $\theta=\frac{\pi}{2}, t=1$

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{2-\cos\theta}=\int_0^1\frac{1}{2-\frac{1-t^2}{1+t^2}}\times\frac{2dt}{1+t^2}$$

$$=\int_0^1\frac{2dt}{1+3t^2}=\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}t\right)\right]_0^1$$

$$=\frac{2}{\sqrt{3}}\times\frac{\pi}{3}=\frac{2\sqrt{3}\pi}{9}$$

(b) $a=\frac{vdv}{dy}=g-kv^2$

$$\int_0^v\frac{vdv}{g-kv^2}=\int_0^h dy$$

$$h=\left[-\frac{1}{2k}\ln\left(g-kv^2\right)\right]_0^v=-\frac{1}{2k}\ln\frac{g-kv^2}{g}$$

$$\therefore \ln\frac{g-kv^2}{g}=-2kh$$

$$\frac{g-kv^2}{g}=e^{-2kh}$$

$$kv^2=g\left(1-e^{-2kh}\right)$$

$$v=\sqrt{\frac{g}{k}\left(1-e^{-2kh}\right)}$$

(c) (i) Let $u=x^n, du=nx^{n-1}dx$

Let $dv=\sqrt{x+3}, v=\frac{2(x+3)\sqrt{x+3}}{3}$

$$I_n=\left[\frac{2x^n(x+3)\sqrt{x+3}}{3}\right]_{-3}^0-\frac{2n}{3}\int_{-3}^0 x^{n-1}(x+3)\sqrt{x+3}dx$$

$$=-\frac{2n}{3}\int_{-3}^0 x^n\sqrt{x+3}dx-2n\int_{-3}^0 x^{n-1}\sqrt{x+3}dx$$

$$=-\frac{2n}{3}I_n-2nI_{n-1}$$

$$\therefore (3+2n)I_n=-6I_{n-1}$$

$$\therefore I_n=\frac{-6n}{3+2n}I_{n-1}$$

(ii) $I_2=\frac{-12}{7}I_1, I_1=\frac{-6}{5}I_0$

$$I_0=\int_{-3}^0\sqrt{x+3}dx=\left[\frac{2}{3}(x+3)\sqrt{x+3}\right]_{-3}^0=2\sqrt{3}$$

$$\therefore I_2=\frac{-12}{7}\times\frac{-6}{5}\times2\sqrt{3}=\frac{144\sqrt{3}}{35}$$

(d) (i) $\Pr(A \text{ wins})=\frac{1}{3}$

(ii) Pr(A and B win at least 1 game each, but C never wins) = Pr (A or B win all, less A wins all, less B wins all) $=\left(\frac{2}{3}\right)^n-2\left(\frac{1}{3}\right)^n$

(iii) Pr(either A or B or C never wins) = Pr(A and B win, C never wins + A and C win, B never wins + B and C win, A never wins + A wins all + B wins all + C wins all)

$$=3\left[\left(\frac{2}{3}\right)^n-2\left(\frac{1}{3}\right)^n\right]+3\left(\frac{1}{3}\right)^n=3\left[\left(\frac{2}{3}\right)^n-\left(\frac{1}{3}\right)^n\right]$$

Pr(each player wins at least 1 game) = 1 − Pr(either A or B or C never wins) $=1-3\left[\left(\frac{2}{3}\right)^n-\left(\frac{1}{3}\right)^n\right]$

$$=1-\frac{2^n}{3^{n-1}}+\frac{1}{3^{n-1}}=\frac{3^{n-1}-2^n+1}{3^{n-1}}$$

Question 15

(a) (i) $Q\left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)=\left(-a\sin\theta, b\cos\theta\right)$

(ii) $Q'\left(a\sin\theta, -b\cos\theta\right)$

$m_{OP}=\frac{b\sin\theta}{a\cos\theta}=\frac{b}{a}\tan\theta, m_{OQ'}=\frac{-b\cos\theta}{a\sin\theta}=-\frac{b}{a}\cot\theta$

$\tan\angle POQ'=\frac{\frac{b}{a}(\tan\theta+\cot\theta)}{1-\frac{b^2}{a^2}}=\frac{ab}{a^2-b^2}(\tan\theta+\cot\theta)$

For $0\le\theta\le\frac{\pi}{2}$, both $\tan\theta$ and $\cot\theta\ge 0, \therefore \tan\theta+\cot\theta$

≥ 2 (since $x+\frac{1}{x}\ge 2$ if $x\ge 0$)

$\therefore \tan\angle POQ'\ge\frac{2ab}{a^2-b^2}$ or $\angle POQ'\ge\tan^{-1}\frac{2ab}{a^2-b^2}$

(b) (i) $\left(\cos\theta+i\sin\theta\right)^8=\cos 8\theta+i\sin 8\theta$

$=\cos^8\theta+i\binom{8}{1}\cos^7\theta\sin\theta-\binom{8}{2}\cos^6\theta\sin^2\theta$

$-i\binom{8}{3}\cos^5\theta\sin^3\theta+\binom{8}{4}\cos^4\theta\sin^4\theta+i\binom{8}{5}\cos^3\theta\sin^5\theta$

$-\binom{8}{6}\cos^2\theta\sin^6\theta-i\binom{8}{7}\cos\theta\sin^7\theta+\sin^8\theta$

$\therefore \sin 8\theta=\binom{8}{1}\cos^7\theta\sin\theta-\binom{8}{3}\cos^5\theta\sin^3\theta$

$+\binom{8}{5}\cos^3\theta\sin^5\theta-\binom{8}{7}\cos\theta\sin^7\theta$

(ii) $\frac{\sin 8\theta}{\sin 2\theta}=\frac{\binom{8}{1}c^7s-\binom{8}{3}c^5s^3+\binom{8}{5}c^3s^5-\binom{8}{7}cs^7}{2sc}$, where

$c=\cos\theta, s=\sin\theta$

$\frac{\sin 8\theta}{\sin 2\theta}=\frac{1}{2}\left[\binom{8}{1}c^6-\binom{8}{3}c^4s^2+\binom{8}{5}c^2s^4-\binom{8}{7}s^6\right]$

$=\frac{1}{2}\left[8c^6-56c^4s^2+56c^2s^4-8s^6\right]$

$=4\left[(1-s^2)^3-7(1-s^2)^2s^2+7(1-s^2)s^4-s^6\right]$

$=4\left[1-3s^2+3s^4-s^6-7s^2+14s^4-7s^6+7s^4-7s^6-s^6\right]$

$=4\left[1-10s^2+24s^4-16s^6\right]$

(c) (i) $x^n-1=(x-1)\left(1+x+x^2+\ldots+x^{n-1}\right)$

$\therefore x^n-1-n(x-1)=(x-1)\left(1+x+x^2+\ldots+x^{n-1}-n\right)$

(ii) If $x\ge 1, x-1\ge 0, 1+x+x^2+\ldots+x^{n-1}\ge n,$

$\therefore (x-1)\left(1+x+x^2+\ldots+x^{n-1}-n\right)\ge 0$

If $0<x<1, x-1<0, 1+x+x^2+\ldots+x^{n-1}<n,$

$\therefore (x-1)\left(1+x+x^2+\ldots+x^{n-1}-n\right)>0$

$\therefore x^n-1-n(x-1)\ge 0$

$\therefore x^n\ge 1+n(x-1)$

(iii) From (ii), $\left(\frac{a}{b}\right)^n\ge 1+n\left(\frac{a}{b}-1\right)$

$a^nb^{1-n}=\left(\frac{a}{b}\right)^n\times b\ge b+n\left(a-b\right)=na+b(1-n)$

Question 16

(a) Let $n=1$, LHS $= x^3-1$

RHS $=(x-1)\left(x^2+x+1\right)=x^3-1=$ RHS, $\therefore$ true.

Assume $x^{3^n}-1=(x-1)\left(x^2+x+1\right)\ldots\left(x^{2\times 3^{n-1}}+x^{3^{n-1}}+1\right)$

RTP $x^{3^{n+1}}-1=(x-1)\left(x^2+x+1\right)\ldots\left(x^{2\times 3^n}+x^{3^n}+1\right)$

LHS $=x^{3\times 3^n}-1=\left(x^{3^n}\right)^3-1$

$=\left(x^{3^n}-1\right)\left(x^{2\times 3^n}+x^{3n}+1\right)$

$=(x-1)\left(x^2+x+1\right)\ldots\left(x^{2\times 3^{n-1}}+x^{3^{n-1}}+1\right)\left(x^{2\times 3^n}+x^{3n}+1\right)$

$=$ RHS.

Hence, it is true for all $n\geq 1$ by the principle of Induction.

(b) (i) $AB \parallel IH$ and $\dfrac{AB}{IH}=\sqrt{2}, \therefore \Delta ABC \;|||\; \Delta IHC, \therefore \dfrac{BC}{HC}=\sqrt{2}$

Similarly, $\Delta ABC \;|||\; \Delta GBF, \therefore \dfrac{BC}{BF}=\sqrt{2}$

$\therefore HC = BF$

$\therefore BH = FC$

$\therefore DY = ZE$, since $DYHB$ and $ZECF$ are parallelograms, as opposite sides are parallel.

(ii) $BH = BC - HC = BC - DE = \sqrt{2}DE - DE$

$= DE\left(\sqrt{2}-1\right)$

$\dfrac{YZ}{BC}=\dfrac{DE-2BH}{BC}=\dfrac{DE-2DE\left(\sqrt{2}-1\right)}{BC}=\dfrac{DE}{BC}\left(3-2\sqrt{2}\right)$

$=\dfrac{3-2\sqrt{2}}{\sqrt{2}}=\dfrac{3\sqrt{2}}{2}-2$

(c) (i) $\left(\beta-\gamma\right)^2=\left(\beta+\gamma\right)^2-4\beta\gamma$

$=\left(\sum\alpha-\alpha\right)^2-4\dfrac{\prod\alpha}{\alpha}$

$=\alpha^2+\dfrac{4q}{\alpha}$, since $\sum\alpha=0, \prod\alpha=-q$

(ii) Let $x=\alpha^2+\dfrac{4q}{\alpha}=\dfrac{\alpha^3+4q}{\alpha}=\dfrac{-p\alpha-q+4q}{\alpha}=-p+\dfrac{3q}{\alpha}$

$\therefore \alpha=\dfrac{3q}{x+p}$

$\therefore q(x)=\left(\dfrac{3q}{x+p}\right)^3+p\left(\dfrac{3q}{x+p}\right)+q=0$ is the equation with roots $\left(\alpha-\beta\right)^2, \left(\alpha-\gamma\right)^2$ and $\left(\beta-\gamma\right)^2$.

$q(x)=27q^3+3pq(x+p)^2+q(x+p)^3$

$=q\left(27q^2+3p(x+p)^2+(x+p)^3\right)$

$\therefore \left(\alpha-\beta\right)^2\left(\alpha-\gamma\right)^2\left(\beta-\gamma\right)^2$ is the product of roots of the above equation, which is $-\left(27q^2+3p\times p^2+p^3\right)$

$=-\left(27q^2+4p^3\right)$

(iii) If $27q^2+4p^3<0$ then $\left(\alpha-\beta\right)^2\left(\alpha-\gamma\right)^2\left(\beta-\gamma\right)^2>0$,

If $p(x)$ has repeated roots then $\left(\alpha-\beta\right)^2\left(\alpha-\gamma\right)^2\left(\beta-\gamma\right)^2$ $=0, \therefore$ the 3 roots must be distinct.

If $p(x)$ has complex roots then the 2 complex roots must be conjugates, since all coefficients are real.

Let $\alpha=a+ib$ and $\beta=\overline{\alpha}=a-ib$.

$\left(\alpha-\beta\right)^2=\left(\alpha-\overline{\alpha}\right)^2=(2ib)^2=-4b^2$

$\therefore \left(\alpha-\beta\right)^2\left(\alpha-\gamma\right)^2\left(\beta-\gamma\right)^2<0.$

$\therefore$ If $\left(\alpha-\beta\right)^2\left(\alpha-\gamma\right)^2\left(\beta-\gamma\right)^2>0$, the 3 roots must be real and distinct.