Multiple Choice

1 C) $x^8 - 1 = 0$, as it has 8 solutions, starting with 1

2 B) ellipse, $PS = \frac{1}{2}PM$

3 D) $3 - i$.

4 C) $y = f^2(x)$

5 B) $\sum\alpha^3 = 2\sum\alpha - 6 = -6$

6 A) $\prod\alpha = 5\alpha = -15, \therefore \alpha = -3, \therefore 9a + 9 = 0, \therefore a = -1$

7 A) $f(x)$ is even, $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$g(x)$ is odd, $\int_{-a}^{a} g(x)dx = 0$

8 B) $f\left(f(-x)\right) = f\left(-f(x)\right) = -f\left(f(x)\right)$

9 C) $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$. When $\frac{dx}{dt} = y, x + \frac{dy}{dt} = 0$

10 B) $f(0) + f(1) \geq 2f\left(\frac{1}{2}\right), \therefore f(x)$ is concave up, $\therefore$ The area enclosed by the curve and the x-axis, $0 \leq x \leq 1$, is less than the rectangle of unit width and height = midpoint of $f(0)$ and $f(1)$.

Question 11

(a) $z = 1 - \sqrt{3}i, w = 1 + i$

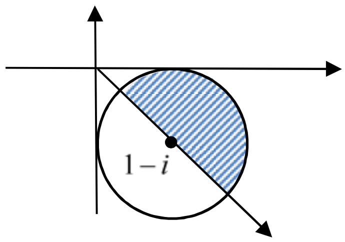
(i) $\arg(z) = -\frac{\pi}{3}$

(ii) $\arg\frac{z}{w} = \arg(z) - \arg(w) = -\frac{\pi}{3} - \frac{\pi}{4} = \frac{-7\pi}{12}$

(b) The equation of the asymptote is $y = \frac{2x}{2\sqrt{3}} = \frac{x}{\sqrt{3}}$

$\therefore \alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

(c)



(d) Let $t = \tan\frac{\theta}{2}, dt = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta, \therefore d\theta = \frac{2dt}{1+t^2}$.

When $\theta = 0, t = 0$. When $\theta = \frac{2\pi}{3}, t = \sqrt{3}$

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}d\theta = \int_0^{\sqrt{3}} \frac{1}{1+\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = \int_0^{\sqrt{3}} dt = \sqrt{3}$$

(e) $\partial V = 2\pi rh\partial y$, where $r = y, h = x_2 - x_1 = (3-y) - \frac{y}{2}$

$= 3 - \frac{3y}{2}. \therefore \partial V = 2\pi\left(3y - \frac{3y^2}{2}\right)\partial y = 3\pi\left(2y - y^2\right)\partial y.$

$\therefore V = 3\pi\int_0^2 \left(2y - y^2\right)dy.$

(f) Let $x = \sin^2\theta, dx = 2\sin\theta\cos\theta d\theta$.

When $x = 0, \theta = 0$. When $x = \frac{1}{2}, \theta = \frac{\pi}{4}$.

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}dx = \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta}2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\sin^2\theta d\theta = \int_0^{\frac{\pi}{4}} (1 - \cos 2\theta)d\theta$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}.$$

Question 12

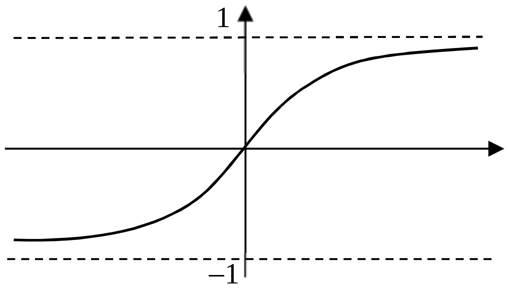
(a) (i) $f'(x)=\dfrac{e^x\left(e^x+1\right)-e^x\left(e^x-1\right)}{\left(e^x+1\right)^2}=\dfrac{2e^x}{\left(e^x+1\right)^2}$

>0 always, $\therefore\ f(x)$ is increasing for all x.

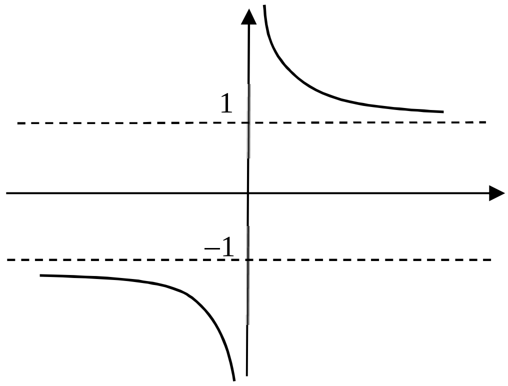
(ii) $f(-x)=\dfrac{e^{-x}-1}{e^{-x}+1}=\dfrac{1-e^x}{1+e^x}=-f(x),\therefore\ f(x)$ is odd

(iii) As $x\to+\infty,\dfrac{e^x-1}{e^x+1}=\dfrac{1-e^{-x}}{1+e^{-x}}\to1^-$.

(iv)



(v)



(b) $z^2+(2+3i)z+(1+3i)=(z+1)(z+1+3i)$

$\therefore z=-1,-1-3i$

(c) $\displaystyle\int x\tan^{-1}x\,dx$

Let $u=\tan^{-1}x, du=\dfrac{1}{1+x^2}dx$, and $dv=xdx, v=\dfrac{x^2}{2}$

$I=\dfrac{x^2}{2}\tan^{-1}x-\dfrac{1}{2}\displaystyle\int\dfrac{x^2}{1+x^2}dx$

$=\dfrac{x^2}{2}\tan^{-1}x-\dfrac{1}{2}\displaystyle\int\left(1-\dfrac{1}{1+x^2}\right)dx$

$=\dfrac{x^2}{2}\tan^{-1}x-\dfrac{x}{2}+\dfrac{1}{2}\tan^{-1}x+C.$

(d) (i) Let $P(x)=(x-\alpha)^2Q(x)$

$P'(x)=2(x-\alpha)Q(x)+(x-\alpha)^2Q'(x).$

$\therefore P(\alpha)=P'(\alpha)=0.$

(ii) $P'(x)=4x^3-9x^2+2x=x(4x-1)(x-2)$

$P(2)=16-24+4+4=0.$

$\therefore\alpha=2,$ since $P(2)=P'(2)=0.$

Question 13

(a) $(\sqrt{r}-\sqrt{s})^2\ge0,\therefore r+s\ge2\sqrt{rs},\therefore\dfrac{r+s}{2}\ge\sqrt{rs}$

(b) (i) $P(\alpha)=\alpha^4+a\alpha^3+b\alpha^2+c\alpha+1=0$

$P\left(\dfrac{1}{\alpha}\right)=\dfrac{1}{\alpha^4}+\dfrac{a}{\alpha^3}+\dfrac{b}{\alpha^2}+\dfrac{c}{\alpha}+1$

$=\dfrac{1+a\alpha+b\alpha^2+c\alpha^3+\alpha^4}{\alpha^4}=0,$ where $\alpha\ne0$

$\therefore\alpha^4+a\alpha^3+b\alpha^2+c\alpha+1=1+a\alpha+b\alpha^2+c\alpha^3+\alpha^4$

$\therefore a=c$

(ii) $\sum\alpha\beta=\alpha\beta+\dfrac{1}{\alpha\beta}+\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}+2=b.$

From (a), $\alpha\beta+\dfrac{1}{\alpha\beta}\ge2,\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\ge2,\therefore b\ge6.$

(c) $v\dfrac{dv}{dx}=-g-kv^2$

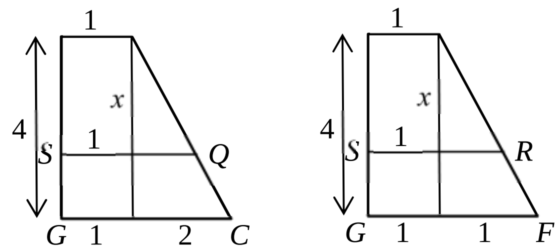
$\displaystyle\int_{\frac{1}{2}\sqrt{\frac{g}{k}}}^{0}\frac{-vdv}{g+kv^2}=\int_0^H dx$

$H=\dfrac{1}{2k}\left[\ln(g+kv^2)\right]_0^{\frac{1}{2}\sqrt{\frac{g}{k}}}$

$=\dfrac{1}{2k}\ln\dfrac{g+k\left(\dfrac{g}{4k}\right)}{g}=\dfrac{1}{2k}\ln\dfrac{5}{4}$

(d) Area$\left(RPQ\right)=SQ\times RS,$ where $S=(x,0).$

By similar triangles, where $G=(4,0)$



$\dfrac{SQ-1}{2}=\dfrac{RS-1}{1}=\dfrac{x}{4},\therefore SQ=\dfrac{x}{2}+1, RS=\dfrac{x}{4}+1$

$\therefore$ Area $=\left(\dfrac{x}{2}+1\right)\left(\dfrac{x}{4}+1\right)=\dfrac{x^2}{8}+\dfrac{3x}{4}+1.$

Volume $=\displaystyle\int_0^4\left(\frac{x^2}{8}+\frac{3x}{8}+1\right)dx=\left[\frac{x^3}{24}+\frac{3x^2}{8}+x\right]_0^4=\frac{38}{3}$ u^3

(e) $\overrightarrow{DC}=\overrightarrow{DA}$ rotated $(-90°)$

$c-d=(a-d)(-i)$

$\therefore c=(1+i)d-ia.$

Question 14

(a) (i) By equating the coefficients of $x, A-B=0$

By equating the constants, $A+B=8$.

$\therefore A=B=4$.

(ii) $\int_0^m \frac{16}{x^4+4}dx = \int_0^m \frac{2x+4}{x^2+2x+2}dx + \int_0^m \frac{-2x+4}{x^2-2x+2}dx$

$= \int_0^m \frac{2x+2}{x^2+2x+2}dx - \int_0^m \frac{2x-2}{x^2-2x+2}dx$

$+\int_0^m \frac{2}{(x+1)^2+1}dx + \int_0^m \frac{2}{(x-1)^2+1}dx$

$= \left[\ln\frac{x^2+2x+2}{x^2-2x+2} + 2\tan^{-1}(x+1) + 2\tan^{-1}(x-1) \right]_0^m$

$= \ln\frac{m^2+2m+2}{m^2-2m+2} + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1) + \frac{\pi}{4} - \frac{\pi}{4}$

$= \ln\frac{m^2+2m+2}{m^2-2m+2} + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1).$

(iii) As $m \to \infty, \ln\frac{m^2+2m+2}{m^2-2m+2} \to \ln 1 = 0,$

and $\tan^{-1}(m \pm 1) \to \frac{\pi}{2},$

$\therefore \lim_{m\to\infty} \int_0^m \frac{16}{x^4+4}dx = 2\pi$

(b) (i) angles subtending the same arc are equal

(ii) $\angle EDA = \angle EBA$ (angles subtending the same arc)

$\angle EBA = AFG$ (same reason)

$\therefore \angle EDA = \angle AFC.$

(iii) Let $\angle DBE = \alpha, \angle EBA = \beta.$

$\angle GEF = 180° - \angle DBA = 180° - (\alpha + \beta)$

(opposite angles in a cyclic quad are supplementary)

$\therefore \angle EGC = \alpha$ (angle sum in ΔEGF)

$\therefore \angle EGC = \angle CBD$ (both $= \alpha$)

$\therefore BCGD$ is a cyclic quad (the interior angle equals the opposite interior angle)

(c) (i) V: $R\sin\theta = mg$ (1)

H: $R\cos\theta = mrw^2$ (2)

$\frac{(2)}{(1)} \Rightarrow \frac{rw^2}{g} = \cot\theta,$ but $\cot\theta = \frac{h}{r}$

$\therefore w^2 = \frac{gh}{r^2}.$

(ii) V: $T\cos\theta + N\sin\theta = mg$ (3)

H: $T\sin\theta - N\cos\theta = mrw^2$ (4)

$(3)\times\sin\theta - (4)\times\cos\theta$ gives

$N = m\left(g\sin\theta - rw^2\cos\theta\right) = m\left(g\sin\theta - r\frac{gh}{r^2}\cos\theta\right)$

$= mg\left(\sin\theta - \frac{h}{r}\cos\theta\right).$

(iii) $N \geq 0, \therefore \tan\theta \geq \frac{h}{r}$, but $\frac{h}{r} = \cot\theta = \frac{1}{\tan\theta}.$

$\therefore \tan^2\theta \geq 1$

$\therefore \theta \geq \frac{\pi}{4}$

Question 15

(a) (i) $I_1 = \int_0^1 x\sqrt{1-x^2}\,dx = \left[\frac{-\sqrt{(1-x^2)^3}}{3}\right]_0^1 = \frac{1}{3}$

(ii) Let $u = x^{n-1}, dv = x\sqrt{1-x^2}\,dx$

$du = (n-1)x^{n-2}, v = \frac{-\sqrt{(1-x^2)^3}}{3}.$

$$I_n = \left[\frac{-x^{n-1}\sqrt{(1-x^2)^3}}{3}\right]_0^1 + (n-1)\int_0^1 x^{n-2}\frac{\sqrt{(1-x^2)^3}}{3}dx$$

$$= \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx$$

$$= \frac{n-1}{3}\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx - \frac{n-1}{3}\int_0^1 x^n\sqrt{1-x^2}\,dx$$

$$= \frac{n-1}{3}I_{n-2} - \frac{n-1}{3}I_n.$$

$\therefore (n+2)I_n = (n-1)I_{n-2}.$

$\therefore I_n = \left(\frac{n-1}{n+2}\right)I_{n-2}.$

(iii) $I_5 = \frac{4}{7}I_3, I_3 = \frac{2}{5}I_1$

$\therefore I_5 = \frac{4}{7}\times\frac{2}{5}\times\frac{1}{3} = \frac{8}{105}.$

(b) (i) By implicit differentiation,

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0, \therefore \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$

At point $(c,d), m = -\frac{\sqrt{d}}{\sqrt{c}}.$

Equation of tangent at point (c,d)

$y - d = -\frac{\sqrt{d}}{\sqrt{c}}(x-c)$

$y\sqrt{c} + x\sqrt{d} = d\sqrt{c} + c\sqrt{d}.$

(ii) Let $y = 0, x = c + \sqrt{cd}, \therefore A\left(c+\sqrt{cd}, 0\right)$

Let $x = 0, y = d + \sqrt{cd}, \therefore B\left(0, d+\sqrt{cd}\right)$

$OA + OB = c + d + 2\sqrt{cd} = \left(\sqrt{c}+\sqrt{d}\right)^2 = a$

(c) (i) $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{x_1^2}{c^2} - \frac{y_1^2}{d^2}$

$y_1^2\left(\frac{1}{b^2} + \frac{1}{d^2}\right) = x_1^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right)$

$y_1^2\left(\frac{b^2+d^2}{b^2d^2}\right) = x_1^2\left(\frac{a^2-c^2}{a^2c^2}\right)$

$\frac{x_1^2}{y_1^2} = \frac{a^2c^2}{(a^2-c^2)}\frac{(b^2+d^2)}{b^2d^2}$

(ii) The x-coordinate of the positive focus is $\sqrt{a^2-b^2}$ for the ellipse and $\sqrt{c^2+d^2}$ for the hyp.

If $a^2 - b^2 = c^2 + d^2$ then $a^2 - c^2 = b^2 + d^2$

$\therefore \frac{x_1^2}{y_1^2} = \frac{a^2c^2}{b^2d^2}.$ (1)

For the ellipse, $m_1 = -\frac{b^2x_1}{a^2y_1}.$

For the hyperbola, $m_2 = \frac{d^2x_1}{c^2y_3}.$

$m_1m_2 = -\frac{b^2d^2x_1^2}{a^2c^2y_1^2} = -1,$ using (1)

Question 16

(a) (i) $\alpha^k = \cos k\theta + i\sin k\theta,$

$\alpha^{-k} = \cos(-k\alpha) + i\sin(-k\alpha) = \cos k\alpha - i\sin k\alpha$

$\therefore \alpha^k + \alpha^{-k} = 2\cos k\alpha.$

(ii) This is a GP, with $a = \alpha^{-n}, r = \alpha$ and $2n+1$ terms

$$C = \frac{\alpha^{-n}(1-\alpha^{2n+1})}{1-\alpha} = \frac{\alpha^{-n}(1-\alpha^{2n+1})(1-\bar{\alpha})}{(1-\alpha)(1-\bar{\alpha})}$$

$$= \frac{(\alpha^{-n}-\alpha^{n+1})(1-\alpha^{-1})}{(1-\alpha)(1-\bar{\alpha})} = \frac{\alpha^{-n}-\alpha^{n+1}-\alpha^{-n-1}+\alpha^{n}}{(1-\alpha)(1-\bar{\alpha})}$$

$$= \frac{\alpha^{n}+\alpha^{n}-\left(\alpha^{n+1}+\alpha^{-(n+1)}\right)}{(1-\alpha)(1-\bar{\alpha})}$$

(iii) Rearranging the series

$$C = 1+\left(\alpha+\alpha^{-1}\right)+\left(\alpha^2+\alpha^{-2}\right)+\ldots+\left(\alpha^n+\alpha^{-n}\right)$$

$= 1+2\left(\cos\theta+\cos 2\theta+\ldots+\cos n\theta\right)$, from (i)

$= \dfrac{2\cos n\theta - 2\cos(n+1)\theta}{1+(\alpha\bar{\alpha})-(\alpha+\bar{\alpha})}$, from (ii)

$$= \frac{2\cos n\theta - 2\cos(n+1)\theta}{2-2\cos\theta}$$

$$= \frac{\cos n\theta - \cos(n+1)\theta}{1-\cos\theta}$$

(iv) $\cos\theta + \cos 2\theta + \ldots + \cos n\theta$

$$= \frac{1}{2}\left(\frac{\cos n\theta - \cos(n+1)\theta}{1-\cos\theta} - 1\right)$$

Let $\theta = \dfrac{\pi}{n}$,

$$\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \ldots + \cos\frac{n\pi}{n}$$

$$= \frac{1}{2}\left(\frac{\cos\frac{n\pi}{n} - \cos\frac{(n+1)\pi}{n}}{1-\cos\frac{\pi}{n}} - 1\right)$$

$$= \frac{1}{2}\left(\frac{-1-\cos\left(\pi+\frac{\pi}{n}\right)}{1-\cos\frac{\pi}{n}} - 1\right)$$

$$= \frac{1}{2}\left(\frac{-1+\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}} - 1\right) = \frac{1}{2}\times -2 = -1$$

(b) Given $e = 2$ and $|\pm ae - a| = 1$,

$\therefore 2a - a = 1, \therefore a = 1$

or $|-2a - a| = 3a = 1, \therefore a = \dfrac{1}{3}$

(c) (i) Given that tile A has x ways and tile B has $x-1$ ways, if tile C is the same colour as tile B (1 colour is used), then tile D has $x-1$ colours to choose from as it can have the same colour as tile A;

if tile C is a different colour from tile B (i.e. both A and B, 2 colours are used) then tile C can have $x-2$ ways and tile D can have $x-2$ ways as it cannot have the same colour as tiles B and C but it can have the same colour as tile A.

$\therefore 1\times(x-1)+(x-2)^2 = x^2-3x+3$

(ii) Let $n = 1$, the number of ways is $x(x-1)$, as the first tile can be painted in x ways and the second tile can be painted in $x-1$ ways, $\therefore$ total $= x(x-1)$ ways.

Assume the 2 by n grid can be painted by

$x(x-1)(x^2-3x+3)^n$ ways.

RTP the 2 by $n+1$ grid can be painted by

$x(x-1)(x^2-3x+3)^{n-1}$ ways.

By adding an extra column, those 2 tiles can have x^2-3x+3 ways, as argued in part (i).

$\therefore$ Total $= x(x-1)(x^2-3x+3)^{n-1}(x^2-3x+3)$

$= x(x-1)(x^2-3x+3)^n$ ways.

$\therefore$ True by PMI.

(iii) Let $x = 3, n = 5$, if not all colours are used, total

$= 3\times 2\times\left(3^2-3\times 3+3\right)^4 = 486$ ways. It can be done

using only 2 colours, e.g. $\begin{matrix} W & B & W & B & W \\ B & W & B & W & B \end{matrix}$.

To use all 3 colours, we take away the number of cases where only 2 colours are used. Since there are 3 colours, choose ${}^3C_2$ colours, e.g. W and B, and the 1st grid can be either W or B, $\therefore 3\times 2 = 6$ ways.

$\therefore$ Total $= 486 - 6 = 480$ ways.