

Multiple Choice

- 1) (A) $(-3)^3 + 18 = -9$
 2) (C) $\frac{dN}{dt} = k \times 80e^{kt} = 80(N - 100)$
 3) (B), $x(x+3) = 4 \times 10, x^2 + 3x - 40 = (x+8)(x-5), \therefore = 5$
 4) (C), ${}^{12}C_8 \times {}^4C_1$
 5) (A), $x = -1, -2, y = 0$
 6) (D) $-1 \leq 2x \leq 1, \therefore -\frac{1}{2} \leq x \leq \frac{1}{2}$
 7) (B) $\sin^{-1} \frac{k}{2} = \frac{\pi}{3}, \therefore \frac{k}{2} = \frac{\sqrt{3}}{2}, \therefore k = \sqrt{3}$
 8) (B), $\lim_{x \rightarrow 3} \frac{\sin(x-3)}{x-3} = 1, \therefore \lim_{x \rightarrow 3} \frac{\sin(x-3)}{(x-3)(x+2)} = \frac{1}{5}$
 9) (D), $a = 6, T = \frac{2\pi}{n} \alpha \frac{1}{n}, \therefore n = 8$
 10) (C), $\cos\left(2t - \frac{\pi}{3}\right) = 0, \therefore 2t - \frac{\pi}{3} = \frac{\pi}{2} + \pi$
 $2t = \frac{5\pi}{6} + \pi = \frac{11\pi}{6}, \therefore t = \frac{11\pi}{12}$

Question 11

- (a) $\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$
 (b) $\tan \alpha = \left| \frac{2 - (-3)}{1 + 2 \times (-3)} \right| = 1, \therefore \alpha = 45^\circ$
 (c) $\frac{4}{x+3} \geq 1$
 $4(x+3) \geq (x+3)^2$
 $(x+3)(4-x-3) \geq 0$
 $(x+3)(1-x) \geq 0$
 $\therefore -3 < x \leq 1$
 (d) $5 \cos x - 12 \sin x = 13 \cos\left(x + \tan^{-1} \frac{12}{5}\right)$
 $\approx 13 \cos(x + 1.176)$
 (e) $u = 2x - 1, du = 2dx$ and $x = \frac{u+1}{2}$
 $\int_1^2 \frac{x}{(2x-1)^2} du = \frac{1}{4} \int_1^3 \frac{(u+1)}{u^2} du$
 $= \frac{1}{4} \int_1^3 \left(\frac{1}{u} + \frac{1}{u^2}\right) du$
 $= \frac{1}{4} \left[\ln u - \frac{1}{u} \right]_1^3$
 $= \frac{1}{4} \left(1 - \frac{1}{3} + \ln 3 \right)$
 $= \frac{1}{4} \left(\frac{2}{3} + \ln 3 \right)$
 (f) (i) $P(3) = 0, \therefore 27 - 9k + 15 + 12 = 0, \therefore k = 6$
 (ii) $x^3 - 6x^2 + 5x + 12 = (x-3)(x^2 - 3x - 4)$
 $= (x-3)(x-4)(x+1)$
 $\therefore x = 3, 4, -1.$

Question 12

- (a) (i) $\angle ACB = 90^\circ - 30^\circ = 60^\circ$ (semi-circle angle = 90°)
 (ii) $\angle ADX = 30^\circ$ (angle between the tangent and a chord is equal angle in the alternate segment)
 (iii) $\angle CDB = 70^\circ$ (exterior angle in a triangle is equal to the sum of two opposite interior angles)
 $\angle CAB = 70^\circ$ (angles subtending the same arc are equal)

(b) (i) Sub $(0, a)$ to the equation of PQ ,

$$-2a - 2apq = 0$$

If this statement is true then $pq = -1$.

\therefore If PQ is a focal chord then $pq = -1$.

(ii) $8a = 2ap, \therefore p = 4$

$$\therefore q = -\frac{1}{4}, \therefore Q(2aq, aq^2) \text{ becomes } \left(-\frac{a}{2}, \frac{a}{16}\right).$$

(c) (i) $\tan 15^\circ = \frac{h}{OA}, \therefore OA = h \cot 15^\circ$.

(ii) Similarly, $OB = h \cot 13^\circ$

$$AB^2 = OB^2 - OA^2$$

$$2000^2 = h^2 (\cot^2 13^\circ - \cot^2 15^\circ)$$

$$\therefore h = \frac{2000}{\sqrt{\cot^2 13^\circ - \cot^2 15^\circ}} = 909.7 \text{ m}$$

(d) (i) By the Cosine rule,

$$160^2 = r^2 + r^2 - 2r \times r \times \cos \theta = 2r^2 (1 - \cos \theta)$$

(ii) Arc length $200 = r\theta, \therefore r = \frac{200}{\theta}$

$$\therefore 160^2 = 2 \times \frac{200^2}{\theta^2} (1 - \cos \theta)$$

$$8\theta^2 = 25(1 - \cos \theta)$$

$$\therefore 8\theta^2 + 25 \cos \theta - 25 = 0$$

(iii) $f(\theta) = 8\theta^2 + 25 \cos \theta - 25$

$$f'(\theta) = 16\theta - 25 \sin \theta$$

$$\therefore \theta_1 = \pi - \frac{8\pi^2 + 25 \cos \pi - 25}{16\pi - 25 \sin \pi}$$

$$= \pi - \frac{8\pi^2 - 50}{16\pi} = 2.57 \text{ (to 2 dp)}$$

Question 13

(a) (i) $x = 3$ or 7 m

(ii) $\sqrt{11}$ m/s

(iii) $a = \text{amplitude} = 2$

$c = \text{centre} = 5$

$$\text{When } x = 5, v^2 = 11, \therefore 11 = 4n^2, \therefore n = \frac{\sqrt{11}}{2}$$

(b) (i) $\left(2x + \frac{1}{3x}\right)^{18} = \frac{(6x^2 + 1)^{18}}{(3x)^{18}}$.

Coefficient of $x^{14} = \text{coefficient of } (x^2)^{16} \text{ in } (6x^2 + 1)^{18}$

$$\therefore \text{It's } = \frac{{}^{18}C_{16} 6^{16}}{3^{18}} = \frac{{}^{18}C_{16} 2^{16}}{3^2}.$$

(ii) The term independent of $x = \text{coefficient of } (x^2)^9$ in $(6x^2 + 1)^{18}$.

$$\therefore \text{It's } = \frac{{}^{18}C_9 6^9}{3^{18}} = \frac{{}^{18}C_9 2^9}{3^9}$$

(c) Let $n = 1$, LHS = $\frac{1}{2}$, RHS = $1 - \frac{1}{2} = \frac{1}{2} \therefore$ It's true

for $n = 1$.

$$\text{Assume } \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

$$\text{RTP } \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

$$\text{LHS} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$$

$$= 1 - \frac{(n+2) - (n+1)}{(n+2)!}$$

$$= 1 - \frac{1}{(n+2)!} = \text{RHS.}$$

\therefore It's true for all integers $n \geq 1$.

(d) (i) $f'(x) = \frac{-1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-(-x)^2}} \times \frac{d}{dx}(-x)$

$$= \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0, \therefore f(x) \text{ is a constant.}$$

$$\text{Let } x = 0, f(0) = \cos^{-1} 0 + \cos^{-1} 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

$$\therefore \cos^{-1} x + \cos^{-1}(-x) = \pi$$

$$\therefore \cos^{-1}(-x) = \pi - \cos^{-1} x$$

Question 14

(a) (i) When $y = 0$, $Vt \sin \theta - \frac{1}{2}gt^2 = 0$, $\therefore t = \frac{2V \sin \theta}{g}$

Sub to x , $x = V \frac{2V \sin \theta}{g} \cos \theta = \frac{V^2 \sin 2\theta}{g}$.

(ii) $\dot{x} = V \cos \theta = V \cos \frac{\pi}{3} = \frac{V}{2}$

$\dot{y} = V \sin \theta - gt = V \sin \frac{\pi}{3} - gt = \frac{\sqrt{3}V}{2} - g \frac{2V}{\sqrt{3}g} = \frac{-V}{2\sqrt{3}}$

$\frac{\dot{y}}{\dot{x}} = \frac{-V}{\frac{V}{2}} = -2$, \therefore Angle = $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(iii) The negative sign above indicates that the particle is travelling downwards.

(b) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = x - 1$

$\frac{1}{2}\dot{x}^2 = \frac{x^2}{2} - x + C$

When $x = 0$, $\dot{x} = 1$, $\therefore C = \frac{1}{2}$.

$\therefore \frac{1}{2}\dot{x}^2 = \frac{x^2}{2} - x + \frac{1}{2}$

$\therefore \dot{x}^2 = x^2 - 2x + 1 = (x-1)^2$

$\therefore \dot{x} = \pm(x-1)$.

When $x = 0$, $\dot{x} = 1$, \therefore take $\dot{x} = 1 - x$.

(ii) $\dot{x} = \frac{dx}{dt} = 1 - x$

$\frac{dt}{dx} = \frac{1}{1-x}$

$t = -\ln(1-x) + C$

$t = 0$, $x = 0$, $\therefore C = 0$

$\therefore t = -\ln(1-x)$

$\therefore 1-x = e^{-t}$

$\therefore x = 1 - e^{-t}$

(iii) When $t \rightarrow \infty$, $x \rightarrow 1$.

The limiting position is $x = 1$.

(c) (i) $\Pr(\text{win 5 in 7 games}) = \Pr(\text{win 4 in 6 games and$

$\text{win the last game}) = \binom{6}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 \times \frac{1}{2}$

$= \binom{6}{4} \left(\frac{1}{2}\right)^7$

(ii) $\Pr(\text{win 5 in at most 7 games}) = \Pr(\text{win 5 in 5 games}) + \Pr(\text{win 5 in 6 games}) + \Pr(\text{win 5 in 7 games})$

$= \binom{4}{4} \left(\frac{1}{2}\right)^5 + \binom{5}{4} \left(\frac{1}{2}\right)^6 + \binom{6}{4} \left(\frac{1}{2}\right)^7$

(iii) $\Pr(\text{one player wins } (n+1) \text{ games})$

$= \binom{n}{n} \left(\frac{1}{2}\right)^{n+1} + \binom{n+1}{n} \left(\frac{1}{2}\right)^{n+2} + \binom{n+2}{n} \left(\frac{1}{2}\right)^{n+3} + \dots$

$+ \binom{2n}{n} \left(\frac{1}{2}\right)^{2n+1}$, as after $2n+1$ games there must be

a winner,

$= \frac{1}{2}$, as each player has an equal chance.

Multiplying both sides by 2^{2n+1} ,

$\binom{n}{n} 2^n + \binom{n+1}{n} 2^{n-1} + \binom{n+2}{n} 2^{n-2} + \dots + \binom{2n}{n} = 2^{2n}$.