## Multiple choice questions

1) (D), $2z+\overline{w}=2(5-i)+2-3i=10-2i+2-3i$
$=12-5i$

2) (D), $3x^2-3y^2y'+3xy'+3y=0$,
$\therefore y'=\dfrac{x^2+y}{y^2-x}=\dfrac{1+2}{4-1}=1$

3) (A), $\overline{z}$ is symmetrical with $z$ about the $x$-axis,
$i\overline{z}$ is $\overline{z}$ rotated 90° anticlockwise

4) (A), as (B) has sharp points, and (C) and (D) have $x$-intercepts changed.

5) (C), $\alpha\beta\gamma=\dfrac{1}{2}$, so $\dfrac{1}{(\alpha\beta\gamma)^3}=8$

6), (B), $b^2=a^2(e^2-1), \therefore e^2=1+\dfrac{b^2}{a^2}=1+\dfrac{4}{6}=\dfrac{5}{3}$
$=\dfrac{15}{9}, \therefore e=\dfrac{\sqrt{15}}{3}$

7) (A), Vertically, $T\cos\alpha+N=mg$, Horizontally, $T\sin\alpha=mr\omega^2$

8) (B), $P'(x)$ has a double root at 1, and $=0$ when $x=-\dfrac{5}{4}$, so $P(x)$ would have a triple root at $x=1$ and a turning point at $x=-\dfrac{5}{4}$, hence, $x$-intercept $<-\dfrac{5}{4}$

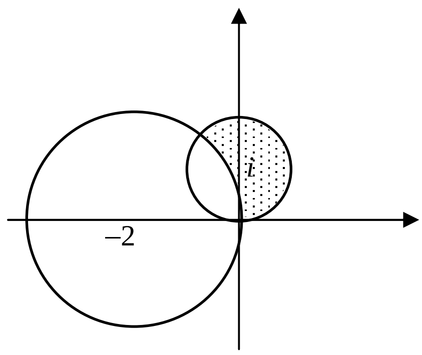
9) (C), $V=2\pi\int_0^2 Rh\,dx$, where $R=2+x, h=y$

10), (B), as the functions in (A) and (D) are odd, $\int_{-a}^{a} f(x)dx=0$,(C) is even but negative, only (B) is even and positive.

## Question 11

(a) $\dfrac{2\sqrt{5}+i}{\sqrt{5}-i}=\dfrac{2\sqrt{5}+i}{\sqrt{5}-i}\times\dfrac{\sqrt{5}+i}{\sqrt{5}+i}=\dfrac{10-1+3\sqrt{5}i}{6}=\dfrac{3+\sqrt{5}i}{2}$

(b)



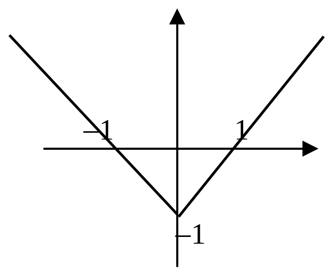

(c) $\displaystyle\int\frac{dx}{x^2+4x+5}=\int\frac{dx}{(x+2)^2+1}=\tan^{-1}(x+2)+C$

(d) (i) $z=2\,\text{cis}\left(-\dfrac{\pi}{6}\right)$

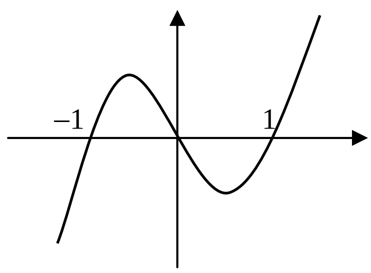
(ii) $z^9=2^9\,\text{cis}\left(\dfrac{-3\pi}{2}\right)=512i$

(e) $\displaystyle\int_0^1\frac{e^{2x}}{e^{2x}+1}dx=\frac{1}{2}\left[\ln\left(e^{2x}+1\right)\right]_0^1=\frac{1}{2}\ln\frac{e^2+1}{2}.$

(f) (i)




(ii)



## Question 12

(a) $t=\tan\frac{\theta}{2}, dt=\frac{1}{2}\sec^2\frac{\theta}{2}d\theta, \therefore d\theta=\frac{2dt}{\sec^2\frac{\theta}{2}}=\frac{2dt}{1+t^2}$

$$\int\frac{d\theta}{1-\cos\theta}=\int\frac{1}{1-\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2}=\int\frac{dt}{t^2}=-\frac{1}{t}=-\cot\frac{\theta}{2}+C.$$

Alternatively, $1-\cos\theta=2\sin^2\frac{\theta}{2}$.

$$\therefore\int\frac{d\theta}{2\sin^2\frac{\theta}{2}}=\int\operatorname{cosec}^2\frac{\theta}{2}d\frac{\theta}{2}=-\cot\frac{\theta}{2}+C.$$

(b) (i) $\frac{2x}{a^2}+\frac{2yy'}{b^2}=0, \therefore y'=-\frac{b^2x}{a^2y}, \therefore m=-\frac{b^2x_0}{a^2y_0}$.

The equation of the tangent:

$$y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0) \quad (1)$$

(ii) The equation of the normal is

$$y-y_0=\frac{a^2y_0}{b^2x_0}(x-x_0)$$

Put $y=0, -b^2x_0=a^2(x-x_0)=a^2x-a^2x_0$

$\therefore x=\frac{(a^2-b^2)x_0}{a^2}=e^2x_0$, since $a^2-b^2=a^2e^2$.

(iii) From (1), let $y=0, a^2y_0{}^2=b^2x_0x-b^2x_0{}^2$.

$\therefore x=\frac{a^2y_0{}^2+b^2x_0{}^2}{b^2x_0}=\frac{a^2b^2}{b^2x_0}$, since $(x_0,y_0)\in$ ellipse

$\therefore x=\frac{a^2}{x_0}$.

$ON\times OT=e^2x_0\times\frac{a^2}{x_0}=a^2e^2=OS^2$.

(c) Let $u=(\ln x)^n, dv=dx$ then $du=n(\ln x)^{n-1}\frac{1}{x}dx, v=x$.

$$\therefore I_n=\left[x(\ln x)^n\right]_1^{e^2}-n\int_1^{e^2}(\ln x)^{n-1}dx$$

$$=e^2\left[\ln(e^2)^n-0\right]-nI_{n-1}$$

$$=e^22^n-nI_{n-1}.$$

(d) (i) $\overrightarrow{A_1B_1}=\overrightarrow{A_1P}$ rotates 90°

$w_1-u_1=(z-u_1)i, \therefore w_1=u_1+i(z-u_1)$

(ii) Similarly, $w_2=u_2-i(z-u_2)$

Midpoint of $B_1B_2=\frac{w_1+w_2}{2}=\frac{u_1+u_2+i(u_2-u_1)}{2}$,

$\therefore$ The locus of the midpoint is a fixed point.

Note:

1) Only $A_1$ and $A_2$ are fixed. $B_1$ and $B_2$ are not fixed, so any arguments based on the midpoint being the centre of semi-circles are not valid.

2) The diagram is misleading/wrong as $A_1, P$ and $B_2$ are not collinear. Neither are $A_2, P$ and $B_1$.

## Question 13

(a) (i) $\frac{dv}{dt}=\frac{400-v^2}{40}, \therefore \int\frac{dv}{400-v^2}=\frac{1}{40}\int dt.$

$\frac{1}{40}t=\int\frac{dv}{(20-v)(20+v)}=\frac{1}{40}\int\left(\frac{1}{20-v}+\frac{1}{20+v}\right)dt$

$=\frac{1}{40}\ln\frac{20+v}{20-v}+C_1.$

$\therefore t=\ln\frac{20+v}{20-v}+C.$

When $t=0, v=0, \therefore C=\ln 1=0.$

$\therefore t=\ln\frac{20+v}{20-v}.$

$e^t=\frac{20+v}{20-v}.$

$20e^t-20=ve^t+v.$

$v=\frac{20(e^t-1)}{e^t+1}.$

(ii) $v\frac{dv}{dx}=\frac{400-v^2}{40}, \therefore \int\frac{vdv}{400-v^2}=\frac{1}{40}\int dx$

$\frac{1}{40}x=-\frac{1}{2}\ln(400-v^2)+C.$

When $t=0, v=0, x=0, \therefore C=\frac{1}{2}\ln 400.$

$\frac{1}{40}x=\frac{1}{2}\ln\frac{400}{400-v^2}.$

$\therefore x=20\ln\frac{400}{400-v^2}.$

(iii) When $t=4, v=\frac{20(e^4-1)}{e^4+1},$

$x=20\ln\frac{400}{400-\frac{400(e^4-1)^2}{(e^4+1)^2}}=20\ln\frac{(e^4+1)^2}{(e^4+1)^2-(e^4-1)^2}$

$=20\ln\frac{(e^4+1)^2}{4e^4}=40\ln\frac{e^4+1}{2e^2}.$

(b) (i) $\angle PSR=\angle QPS$ (alternate angles on parallel lines)

$\angle PRS=\angle S'PQ$ (corresponding angles on // lines)

But $\angle QPS=\angle S'PQ=\alpha, \therefore \angle PSR=\angle PRS, \therefore PS=PR.$

(ii) $\frac{PR}{QS}=\frac{PS'}{QS'}$ (If two intersecting lines are cut by parallel lines then the line segments cut by the parallel lines are proportional).

But $PR=PS, \therefore \frac{PS}{QS}=\frac{PS'}{QS'}.$

(c) (i) $SP=ePM$, where $M$ is the foot of the perpendicular of $P$ on the directrix

$=e\left(a\sec\theta-\frac{a}{e}\right)=a(e\sec\theta-1).$

(ii) $\frac{SP}{S'P}=\frac{QS}{QS'}$, from part (b)

$\frac{a(e\sec\theta-1)}{a(e\sec\theta+1)}=\frac{ae-x_Q}{x_Q+ae}.$

$x_Qe\sec\theta-x_Q+ae^2\sec\theta-ae=ae^2\sec\theta+ae-x_Qe\sec\theta-x_Q.$

$2x_Qe\sec\theta=2ae.$

$\therefore x_Q=\frac{a}{\sec\theta}.$

(iii) The gradient of $PQ$ is $\frac{b\tan\theta-0}{a\sec\theta-\frac{a}{\sec\theta}}$

$=\frac{b\tan\theta\sec\theta}{a(\sec^2\theta-1)}=\frac{b\tan\theta\sec\theta}{a\tan^2\theta}=\frac{b\sec\theta}{a\tan\theta}=$ the gradient of the tangent at $P$.

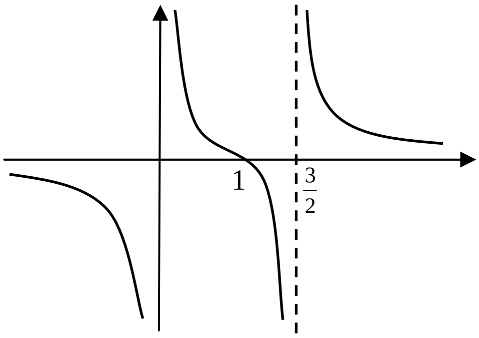
$\therefore PQ$ is the tangent.

## Question 14

(a) $\int \frac{3x^2+8}{x\left(x^2+4\right)}dx = \int \left(\frac{2}{x}+\frac{x}{x^2+4}\right)dx$

$= 2\ln x + \frac{1}{2}\ln(x^2+4) + C.$

(b) (i)




(ii) $\frac{x(2x-3)}{x-1} = \frac{2x^2-3x}{x-1} = \frac{2x(x-1)-(x-1)-1}{x-1}$

$= 2x-1-\frac{1}{x-1}.$

$\therefore$ The line $\ell$ has equation $y = 2x-1$.

(c) Let the dimensions of the rectangular cross-section be $x$ and $y$, and the distance from the back face be $z$.

By similar triangles (figure 1), $\frac{x}{a} = \frac{z}{r}, \therefore x = \frac{az}{r}$.

By Pythagoras' theorem (figure 2), $y = \sqrt{r^2 - z^2}$.

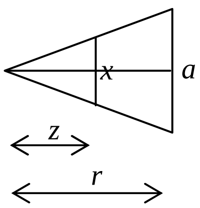



figure 1

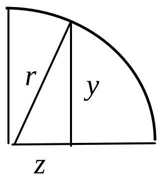



figure 2

$\therefore$ Area $= xy = \frac{a}{r}z\sqrt{r^2-z^2}$.

Volume of cross-section of thickness $\partial z = \frac{a}{r}z\sqrt{r^2-z^2}\,\partial z$.

Volume of solid $= \frac{a}{r}\int_0^r z\sqrt{r^2-z^2}\,\partial z$.

$= \frac{a}{r}\times -\frac{1}{2}\left[\frac{2\sqrt{(r^2-z^2)^3}}{3}\right]_0^r = \frac{ar^2}{3}$ u$^3$.

(d) (i) $\angle GAP = \angle PBF$ (angles in alternate segments)

$\angle G = \angle F = 90°$.

$\therefore \Delta APG \parallel\!| \Delta BPE$ (AA)

(ii) Similarly, $\Delta BPF \parallel\!| \Delta APE$ (AA)

$\frac{PG}{EP} = \frac{AP}{BP}$ (corresponding sides in $\Delta APG$ and $\Delta BPE$)

$\frac{EP}{PF} = \frac{AP}{BP}$ (corresponding sides in $\Delta BPF$ and $\Delta APE$)

$\therefore \frac{PG}{EP} = \frac{EP}{PF}$.

$\therefore EP^2 = PF \times GP$.

## Question 15

(a) (i) $\left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0$

$a+b-2\sqrt{ab} \ge 0.$

$\sqrt{ab} \le \dfrac{a+b}{2}.$

(ii) $y \ge x, \therefore y-x \ge 0.$

$y(x-1)-x(x-1) \ge 0,$ since $x-1 \ge 0$

$xy-x^2+x-y \ge 0$

$x(y-x+1) \ge y.$

Alternatively, $\text{LHS}-\text{RHS} = xy-x^2+x-y$

$= x(y-x)-(y-x) = (x-1)(y-x) \ge 0,$ since $1 \le x \le y.$

(iii) Put $x=j, y=n$ into part (ii),

$j(n-j+1) \ge n$

$\therefore \sqrt{n} \le \sqrt{j(n-j+1)}$

and $\sqrt{j(n-j+1)} \le \dfrac{j+(n-j+1)}{2} = \dfrac{n+1}{2}$, from part (i)

$\therefore \sqrt{n} \le \sqrt{j(n-j+1)} \le \dfrac{n+1}{2}.$

(iv) Let $j=1, \sqrt{n} \le \sqrt{1(n)} \le \dfrac{n+1}{2}$

Let $j=2, \sqrt{n} \le \sqrt{2(n-1)} \le \dfrac{n+1}{2}$

Let $j=3, \sqrt{n} \le \sqrt{3(n-2)} \le \dfrac{n+1}{2}$

...

Let $j=n, \sqrt{n} \le \sqrt{n(1)} \le \dfrac{n+1}{2}$

Multiplying them all, $\left(\sqrt{n}\right)^n \le \sqrt{n!n!} \le \left(\dfrac{n+1}{2}\right)^n$

$\therefore \left(\sqrt{n}\right)^n \le n! \le \left(\dfrac{n+1}{2}\right)^n.$

(b) (i) Since all coefficients are real, the roots occur in conjugates. The conjugate of $i\alpha$ is $-i\overline{\alpha}$ since the conjugate of a product is the product of the conjugates.

(ii) $P(z) = z^2(z^2-2kz+k^2)+(k^2z^2-2kz+1)$

$= z^2(z-k)^2+(kz-1)^2.$

(iii) Since $z^2(z-k)^2+(kz-1)^2$ is the sum of two squares,

$P(z)$ has real zeros when $(z-k)^2=(kz-1)^2$ (see note)

$\therefore z^2-2kz+k^2 = k^2z^2-2kz+1.$

$z^2(1-k^2)-(1-k^2)=0.$

$\left(z^2-1\right)\left(1-k^2\right)=0.$

$\therefore z=\pm 1,$ or $k=\pm 1.$

$\therefore k=\pm 1,$ as we want to find $k$ only.

$\therefore$ When $k=1, P(z) = z^2(z-1)^2+(z-1)^2$

$= (z^2+1)(z-1)^2$

When $k=-1, P(z) = z^2(z+1)^2+(z+1)^2$

$= (z^2+1)(z+1)^2.$

(iv) Product of roots $= \alpha\overline{\alpha}i\alpha(-i\overline{\alpha}) = \left(\alpha\overline{\alpha}\right)^2 = \left(|\alpha|\right)^4$

$= 1\left(=\dfrac{e}{a}\right), \therefore |\alpha| = 1, \therefore$ all zeros have modulus 1.

(v) Sum of roots $= (x+iy)+(x-iy)+(-y+ix)$

$+(-y-ix) = 2x-2y = 2k\left(=-\dfrac{b}{a}\right), \therefore k = x-y.$

(vi) Let $\alpha = x+iy = \cos\theta + i\sin\theta,$ since $|\alpha|=1.$

$\therefore k = \cos\theta - \sin\theta = \sqrt{2}\cos\left(\theta+\dfrac{\pi}{4}\right).$

$\therefore -\sqrt{2} \le k \le \sqrt{2},$ since $-1 \le \cos A \le 1.$

Note:

In part (iii), $z^2=(kz-1)^2$ is ignored because if $z^2$ $=(kz-1)^2$ then $(k^2-1)z^2-2kz+1=0.$

$\therefore P(z) = \left((k^2-1)z^2-2kz+1\right)\left((z-k)^2+1\right)$

$= \left((k^2-1)z^2-2kz+1\right)\left(z^2-2kz+k^2+1\right).$

The coefficient of $z^4$ is $1, \therefore k^2-1=1.$

The constant term is $1, \therefore k^2+1=1.$

No values of $k$ satisfy both these equations.

## Question 16

(a) (i) $\frac{(m+n)!}{m!n!}$.

(ii) Consider three identical separators to divide the 10 coins into 4 boxes, total = 13 items, including 3 identical separators and 10 identical coins. (Refer to my 3u book, Fundamental Mathematics, page 251-2)

$\therefore \frac{13!}{3!10!}$.

(b) (i) Let $A=\tan^{-1}x, B=\tan^{-1}y, \therefore \tan A = x, \tan B = y$

$\tan(\text{LHS}) = \frac{\tan A + \tan B}{1-\tan A \tan B} = \frac{x+y}{1-xy}$.

$\therefore \text{LHS} = \tan^{-1}\frac{x+y}{1-xy}$.

(ii) Let $j=1$, LHS = RHS $= \tan^{-1}\frac{1}{2}$.

Assume $\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\frac{n}{n+1}$.

RTP $\sum_{j=1}^{n+1}\tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\frac{n+1}{n+2}$.

$\text{LHS} = \sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right) + \tan^{-1}\frac{1}{2(n+1)^2}$

$= \tan^{-1}\frac{n}{n+1} + \tan^{-1}\frac{1}{2(n+1)^2}$

$= \tan^{-1}\frac{\frac{n}{n+1}+\frac{1}{2(n+1)^2}}{1-\frac{n}{n+1}\times\frac{1}{2(n+1)^2}}$

$= \tan^{-1}\frac{\frac{2n(n+1)+1}{2(n+1)^2}}{\frac{2(n+1)^3-n}{2(n+1)^3}} = \tan^{-1}\frac{(n+1)(2n^2+2n+1)}{2(n+1)^3-n}$

$= \tan^{-1}\frac{(n+1)(2n^2+2n+1)}{2n^3+6n^2+5n+2}$

$= \tan^{-1}\frac{(n+1)(2n^2+2n+1)}{(n+2)(2n^2+2n+1)}$, by inspection

$= \tan^{-1}\frac{n+1}{n+2} = \text{RHS}$.

By the principle of Math Induction, it's true for all $n \geq 1$.

(iii) $\lim_{n\to\infty}\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{2j^2}\right) = \lim_{n\to\infty}\tan^{-1}\frac{n+1}{n+2} = \frac{\pi}{4}$.

(c) (i) $P(k) = \frac{n}{n}\times\frac{n-1}{n}\times\frac{n-2}{n}\times\ldots\times\frac{n-k+1}{n}\times\frac{{}^kC_1}{n}$

$= \frac{(n-1)!}{n^k}\times\frac{k}{1\times2\times\ldots(n-k)} = \frac{(n-1)!}{n^k}\times\frac{k}{(n-k)!}$.

(ii) If $P(k) \geq P(k-1)$ then

$\frac{(n-1)!}{n^k}\times\frac{k}{(n-k)!} \geq \frac{(n-1)!(k-1)}{n^{k-1}(n-k+1)!}$.

$\frac{k}{n} \geq \frac{k-1}{n-k+1}$.

$nk-k^2+k \geq nk-n$, since all terms are positive.

$k^2-k-n \leq 0$.

(iii) If $\sqrt{n+\frac{1}{4}} > k-\frac{1}{2}, n+\frac{1}{4} > k^2-k+\frac{1}{4}$, on squaring both sides.

$\therefore n > k^2-k$.

$\therefore n > k^2-k+\frac{1}{4}$, since as both $n$ and $k^2-k$ are integers, $n$ must be greater than $k^2-k$ by at least 1.

$\therefore n > \left(k-\frac{1}{2}\right)^2$.

$\therefore \sqrt{n} > k-\frac{1}{2}$.

(iv) Solving $k^2-k-n \leq 0$ for some integers $k$,

$0 < k \leq \frac{1+\sqrt{1+4n}}{2} = \frac{1}{2}+\sqrt{n+\frac{1}{4}}$.

$\therefore k-\frac{1}{2} \leq \sqrt{n+\frac{1}{4}}$.

If $4n+1$ is not a perfect square, $k-\frac{1}{2} < \sqrt{n+\frac{1}{4}}$.

$\therefore k-\frac{1}{2} < \sqrt{n}$, from part (iii)

If $P(k) > P(k-1)$ then $k-\frac{1}{2} < \sqrt{n}, \therefore k < \sqrt{n}+\frac{1}{2}$,

$P(k) > P(k-1) > P(k-2) > \ldots > P(1)$.

If $P(k) < P(k-1)$ then $k-\frac{1}{2} > \sqrt{n}, \therefore k > \sqrt{n}+\frac{1}{2}$.

Replace $k$ by $k+1, P(k) > P(k+1) > P(k+2) > \ldots >$ $P(n)$ when $k+1 > \sqrt{n}+\frac{1}{2}, \therefore k > \sqrt{n}-\frac{1}{2}$.

$\therefore P(k)$ is greatest when $\sqrt{n}-\frac{1}{2} < k < \sqrt{n}+\frac{1}{2}$.

$\therefore P(k)$ is greatest when $k$ is the integer closest to $\sqrt{n}$.

Note: The argument that if $k < \sqrt{n} + \frac{1}{2}, k$ and $n$ are integers, then $P(k) > P(k-1)$ is apt to declare that $P(k)$ is greatest when $k$ is the integer closest to $\sqrt{n}$ . For example, when $n = 2, k < \sqrt{2} + \frac{1}{2} \simeq 1.92$, but not more than 1.92, then $P(k)$ is greatest when $k$ = 1, i.e. $k$ is the integer closest to $\sqrt{2}$ .