Question 1

(a) $x=\dfrac{1\times -1+4\times 9}{1+4}=\dfrac{35}{5}=7$

$y=\dfrac{1\times -2+4\times 3}{1+4}=\dfrac{10}{5}=2, \therefore P(7,2)$

(b) $y=\dfrac{\sin^2 x}{x}$

$\dfrac{dy}{dx}=\dfrac{2x\sin x\cos x-\sin^2 x}{x^2}=\dfrac{\sin x(2x\cos x-\sin x)}{x^2}$

(c) $\dfrac{4-x}{x}<1$

$(4-x)x<x^2$

$4x-2x^2<0$

$2x(2-x)<0$

$\therefore x<0$ or $x>2$

(d) $u=\sqrt{x}, du=\dfrac{1}{2\sqrt{x}}dx.$

When $x=1, u=1$; when $x=4, u=2$.

$\int_1^4 \dfrac{e^{\sqrt{x}}}{\sqrt{x}}dx=2\int_1^2 e^u du=2\left[e^u\right]_1^2=2\left(e^2-e\right).$

(e) $\cos^{-1}\left(-\dfrac{1}{2}\right)=\pi-\cos^{-1}\dfrac{1}{2}=\dfrac{2\pi}{3}.$

(f) $x^2+e\ge e$

$\therefore \ln(x^2+e)\ge \ln e=1.$

$\therefore$ Range: $y\ge 1.$

Question 2

(a) $P(3)=12, \therefore 27-9a+3=12, \therefore a=2$

$\therefore P(-1)=-1-2-1=-4$

$\therefore$ Remainder is -4.

(b) $f(x)=\cos 2x-x, \therefore f\left(\dfrac{1}{2}\right)=0.0403$

$f'(x)=-2\sin 2x-1, \therefore f'\left(\dfrac{1}{2}\right)=-2.6829$

$x_2=\dfrac{1}{2}-\dfrac{0.0403}{-2.6829}=0.5150=0.52$ (2 dp)

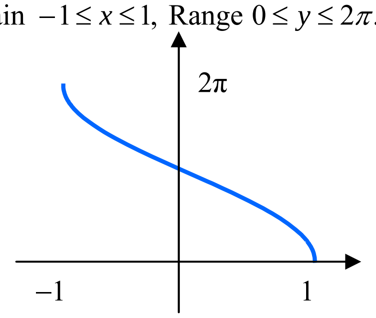
(c) $\left(3x-\dfrac{4}{x}\right)^8=\dfrac{\left(3x^2-4\right)^8}{x^8}.$

$\therefore$ Coefficient of x^2 = coefficient of x^{10} in the expansion of $\left(3x^2-4\right)^8={}^8C_5 3^5(-4)^3$

$=-870912.$

(d) $f(x)=2\cos^{-1}x.$

Domain $-1\le x\le 1$, Range $0\le y\le 2\pi.$



(e) (i) 40!

(ii) 3!37!

Question 3

(a) (i) $\frac{dx}{dt} = -nA\sin nt + nB\cos nt$

$\frac{d^2x}{dt^2} = -n^2A\cos nt - n^2B\sin nt$

$= -n^2\left(A\cos nt + B\sin nt\right) = -n^2x$

(ii) $t = 0, x = 0, \therefore 0 = A\cos n, \therefore A = 0$

$t = 0, \frac{dx}{dt} = 2n, \therefore 2n = nB\cos 0, \therefore B = 2$

(iii) Period $= \frac{2\pi}{n}$, $\therefore$ The time taken to travel from the origin to its max. distance is $\frac{1}{4} \times$ period

$= \frac{\pi}{2n}$.

(iv) Total distance for 1 period $= 4 \times$ amplitude

$= 4 \times 2 = 8$

(b) (i) $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{1} = 2t$

$y - t^2 = 2t(x - t)$

$y = 2tx - 2t^2 + t^2$

$\therefore y = 2tx - t^2$. (1)

(ii) $y = 2(1-t)x - (1-t)^2$ (2)

(iii) (1) = (2) gives $2tx - t^2 = 2(1-t)x - (1-t)^2$

$2tx - t^2 = 2x - 2tx - 1 + 2t - t^2$

$4tx - 2x = -1 + 2t$

$2x(2t - 1) = -1 + 2t$

$2x = 1$

$\therefore x = \frac{1}{2}$

Sub. $x = \frac{1}{2}$ to (1) gives $y = t - t^2$.

$\therefore R\left(\frac{1}{2}, t - t^2\right)$

(iv) The locus of R is the line $x = \frac{1}{2}$.

But $y = t - t^2 = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 \le \frac{1}{4}$.

Since $P \ne Q$, i.e. $t \ne \frac{1}{2}, \therefore y < \frac{1}{4}$.

Question4

(a) (i) $f'(x) = -e^{-x} + 4e^{-2x}$

(ii) $f'(x) = 0$ gives $e^{-x} = 4e^{-2x}$

$1 = 4e^{-x}$

$e^x = 4$

$x = \ln 4$.

$y = e^{-\ln 4} - 2e^{-2\ln 4} = \frac{1}{4} - 2 \times \frac{1}{16} = \frac{1}{8}$.

$\therefore$ SP $\left(\ln 4, \frac{1}{8}\right)$.

Note: The question does not ask you to prove that this is maximum.

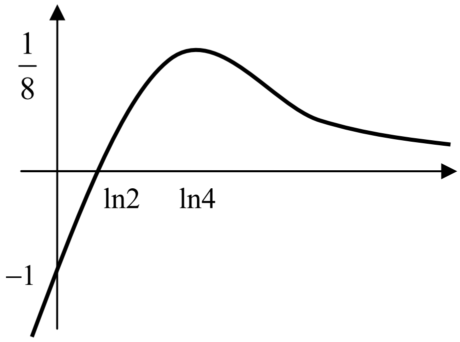
(iii) $f(\ln 2) = e^{-\ln 2} - 2e^{-2\ln 2} = \frac{1}{2} - 2 \times \frac{1}{4} = 0$.

(iv) As $x \to +\infty, e^{-x} \to 0, e^{-2x} \to 0, \therefore y \to 0$

As $x \to -\infty, y \simeq -2e^{-2x} \to -\infty$

(v) When $x = 0, y = 1 - 2 = -1$.

(vi)



(b) (i) Angle at the centre is twice angle on the circumference subtending the same arc.

(ii) $\angle ADC = 2x$ (exterior angle in a Δ = sum of two opposite interior angles)

$\therefore \angle ADC = \angle AOC$.

$\therefore ACDO$ is cyclic (equal angles subtending the same arc)

(iii) $OC = OA$ = radius of circle ABC.

$PA = PC$ = radius of circle $ACDO$.

$\therefore$ Both O and P belong to the perpendicular bisector of AC.

$\therefore O$, P and M are collinear.

Question 5

(a) (i) $\frac{SP}{TQ}=\frac{SN}{TN}$ (corresponding sides of similar Δs)

But $SN=2, \cos\theta = TQ,$ and $\sin\theta = OT$

$=1-TN, \therefore TN = 1-\sin\theta.$

$\therefore \frac{SP}{\cos\theta}=\frac{2}{1-\sin\theta}.$

$\therefore SP=\frac{2\cos\theta}{1-\sin\theta}.$

(ii) $\frac{\cos\theta}{1-\sin\theta}=\frac{\cos\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}$

$=\frac{\cos\theta+\cos\theta\sin\theta}{\cos^2\theta}$

$=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\sec\theta+\tan\theta.$

(iii) $\angle SNP=\frac{1}{2}\angle SOP$ (angle at the circumference is half angle at the centre subtending the same arc)

$=\frac{1}{2}\left(\frac{\pi}{2}+\theta\right)=\frac{\pi}{4}+\frac{\theta}{2}.$

(iv) $\tan\angle SNP=\frac{TQ}{TN}=\frac{\cos\theta}{1-\sin\theta}$

$\therefore \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\sec\theta+\tan\theta.$

(v) $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\sqrt{3}$

$\frac{\pi}{4}+\frac{\theta}{2}=\frac{\pi}{3}+k\pi$

$\frac{\theta}{2}=\frac{\pi}{12}+k\pi$

$\therefore \theta=\frac{\pi}{6}+2k\pi.$

$\therefore$ For $-\frac{\pi}{2}<\theta<\frac{\pi}{2}, \theta=\frac{\pi}{6}.$

(b) (i) $t=1, T=20, \therefore 20=5+25e^{-k}.$

$e^{-k}=\frac{15}{25}=\frac{3}{5}.$

$-k=\ln\frac{3}{5}$

$\therefore k=\ln\frac{5}{3}.$

(ii) $t=0, T=30, A=22, \therefore 30=22+B$

$\therefore B=8.$

$\therefore T=22+8e^{\left(-\ln\frac{5}{3}\right)t}$

$\therefore$ When $T=37, 37=22+8e^{\left(-\ln\frac{5}{3}\right)t}$

$\frac{15}{8}=e^{\left(-\ln\frac{5}{3}\right)t}$

$\ln\frac{15}{8}=-\ln\frac{5}{3}t$

$t=\frac{\ln\frac{15}{8}}{-\ln\frac{5}{3}}=-1.23=-1$ hour 14 minutes ago.

$\therefore$ The object had a temperature of 37° at 08:46.

Question 6

(a) (i) Let $n=1$, $\text{LHS}=1\times 5=5$,

$\text{RHS}=\frac{1}{6}\times 1\times 2\times 15=5.\therefore$ True.

Assume $1\times 5+2\times 6+...+n(n+4)$

$=\frac{1}{6}n(n+1)(2n+13).$

RTP $1\times 5+2\times 6+...+(n+1)(n+5)$

$=\frac{1}{6}(n+1)(n+2)(2n+15).$

$\text{LHS}=\frac{1}{6}n(n+1)(2n+13)+(n+1)(n+5)$

$=\frac{1}{6}(n+1)\left(n(2n+13)+6(n+5)\right)$

$=\frac{1}{6}(n+1)\left(2n^2+13n+6n+30\right)$

$=\frac{1}{6}(n+1)(2n^2+19n+30)$

$=\frac{1}{6}(n+1)(n+2)(2n+15)=\text{RHS}.$

$\therefore$ True for $n+1$.

$\therefore$ True for all $n\geq 1$ by the principle of Induction.

(b) (i) The ball strikes the ground when $y=0$,

$\therefore h=\frac{1}{2}gt^2$

$\therefore t=\sqrt{\frac{2h}{g}}$ s.

(ii) At the moment of impact, $\tan(-45°)=\frac{\dot{y}}{\dot{x}}$

$\therefore -1=\frac{-gt}{v}.$

$\therefore v=gt=g\sqrt{\frac{2h}{g}}=\sqrt{2gh}.$

$\therefore x=\sqrt{2gh}\sqrt{\frac{2h}{g}}=2h.$

$\therefore d=2h.$

(c) (i) Pr(hits target at least once) $=1-$ Pr(hits no target) $=1-(1-p)^2=2p-p^2$.

Alternatively, Pr $=$ Pr(hits target once) $+$ Pr(hits target twice) $={}^2C_1p(1-p)+p^2=2p-p^2$.

(ii) Pr(hits target at least twice) $=$ Pr(hits target twice) $+$ Pr(hits target 3 times)

$={}^3C_2p^2(1-p)+p^3$

$=3p^2-3p^3+p^3=3p^2-2p^3$

(iii) $2p-p^2-\left(3p^2-2p^3\right)=2p-4p^2+2p^3$

$=p\left(p^2-2p+2\right)=p\left((p+1)^2+1\right)>0,$

$\therefore$ Pr(win game 1) $>$ Pr(win game 2)

(iv) $2p-p^2=2\left(3p^2-2p^3\right).$

$2p-p^2=6p^2-4p^3$

$4p^3-7p^2+2p=0$

$p\left(4p^2-7p+2\right)=0$

$\therefore p=\frac{7\pm\sqrt{17}}{8}.$

As $0<p<1$, $p=\frac{7-\sqrt{17}}{8}\simeq 0.36.$

Question 7

(a) (i) Volume remained = Volume of the cone of radius h, height h − Volume of the cone of radius ℓ, height ℓ

$$=\frac{1}{3}\pi h^2\times h-\frac{1}{3}\pi\ell^2\times\ell$$

$$=\frac{\pi}{3}\left(h^3-\ell^3\right)$$

(ii) $$\frac{dV}{dt}=\frac{\pi}{3}\times-3\ell^2\times\frac{d\ell}{dt}$$

$$=-\pi\ell^2\times\frac{d\ell}{dt}$$

$$=-\pi\times4\times10=-40\pi\text{ cm}^3/\text{s}.$$

(iii) When the lower cone has lost $\frac{1}{8}$ of its water,

$$V=\frac{7}{8}\times\frac{\pi}{3}h^3.$$

$$\frac{7}{8}\times\frac{\pi}{3}h^3=\frac{\pi}{3}\left(h^3-\ell^3\right)$$

$$h^3=8\ell^3$$

$$\therefore h=2\ell\text{ or }\ell=\frac{h}{2}.$$

$$\therefore\frac{dV}{dt}=-\pi\ell^2\times\frac{d\ell}{dt}=-\pi\frac{h^2}{4}\times10=-\frac{5\pi h^2}{2}\text{ cm}^3/\text{s}.$$

(b) (i) $$(1+x)^n=\sum_{r=0}^{n}\binom{n}{r}x^r$$

Differentiating both sides wrt x,

$$n(1+x)^{n-1}=\sum_{r=1}^{n}\binom{n}{r}rx^{r-1}.$$

Multiplying both sides by x,

$$nx(1+x)^{n-1}=\sum_{r=1}^{n}\binom{n}{r}rx^r. \qquad (1)$$

(ii) Differentiating (1) again,

$$n(1+x)^{n-1}+n(n-1)x(1+x)^{n-2}=\sum_{r=1}^{n}\binom{n}{r}r^2x^{r-1} \qquad (2)$$

Let $x=1$

$$n2^{n-1}+n(n-1)2^{n-2}=\sum_{r=1}^{n}\binom{n}{r}r^2$$

$$\text{LHS}=2^{n-2}(2n+n^2-n)=2^{n-2}n(n+1)$$

$$\therefore n(n+1)2^{n-2}=\sum_{r=1}^{n}\binom{n}{r}r^2. \qquad (3)$$

(iii) In (2), let $x=-1$,

$$0=\sum_{r=1}^{n}\binom{n}{r}r^2(-1)^{r-1}. \qquad (4)$$

(3) − (4) gives

$$n(n+1)2^{n-2}=\sum_{r=1}^{n}\binom{n}{r}r^2-\sum_{r=1}^{n}\binom{n}{r}r^2(-1)^{r-1}.$$

$$\text{RHS}=\binom{n}{1}1^2+\binom{n}{2}2^2+\binom{n}{3}3^2+\ldots+\binom{n}{n}n^2$$

$$-\binom{n}{1}1^2+\binom{n}{2}2^2-\binom{n}{3}3^2+\ldots+(-1)^n\binom{n}{n}n^2,$$

The last term, due to n being even, $=\binom{n}{n}n^2$.

$$\therefore n(n+1)2^{n-2}=2\left[\binom{n}{2}2^2+\binom{n}{4}4^2+\ldots+\binom{n}{n}n^2\right]$$

$$\therefore\binom{n}{2}2^2+\binom{n}{4}4^2+\ldots+\binom{n}{n}n^2=n(n+1)2^{n-3}.$$