Question 1

(a) $P(-3) = (-3)^3 = -27$.

(b) $\dfrac{-3}{\sqrt{1-9x^2}}$.

(c) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_{-1}^{1} = 2\times\frac{\pi}{6} = \frac{\pi}{3}$.

(d) ${}^{12}C_4 2^8 3^4$.

(e) $\left[\dfrac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{4}} = \dfrac{1}{3}\times\left(\dfrac{\sqrt{2}}{2}\right)^3 = \dfrac{2\sqrt{2}}{24}$.

(f) $(x-3)(5-x) > 0, \therefore 3 < x < 5$.

Question 2

(a) $u = \ln x$, $du = \dfrac{1}{x}dx.$

When $x = e, u = 1$; when $x = e^2, u = 2.$

$\displaystyle\int_1^2 \frac{1}{u^2}du = \left[\frac{-1}{u}\right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}.$

(b) $\dfrac{1}{2}v^2 = \dfrac{x^2}{2} + 4x + C.$

When $x = 1, v = 0, \therefore C = -\dfrac{9}{2}.$

$\therefore v^2 = x^2 + 8x - 9.$

When $x = 2, v^2 = 11, \therefore$ Speed $= \sqrt{11}$ m/s.

(c) $\sum\alpha = -2 + 3 + \alpha = \dfrac{-16}{a}.$

$1 + \alpha = -\dfrac{16}{a}$ (1)

$\prod\alpha = -6\alpha = \dfrac{120}{a}. \therefore a = -\dfrac{20}{\alpha}.$

From (1), $1 + \alpha = \dfrac{16}{20}\alpha = \dfrac{4}{5}\alpha.$

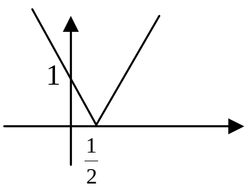
$1 = -\dfrac{1}{5}\alpha. \therefore \alpha = -5.$

(d) $f'(x) = \sec^2 x - \dfrac{1}{x}.$

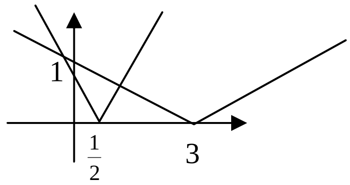
$x_1 = 4 - \dfrac{\tan 4 - \ln 4}{\sec^2 4 - \frac{1}{4}} = 4.11$.

Question 3

(a)



(b)



From the graph, $|2x-1| \le |x-3|$ for $\pm(2x-1) \le -x+3$.

$2x - 1 \le -x + 3$ gives $3x \le 4, \therefore x \le \dfrac{4}{3}.$

$-2x + 1 \le -x + 3$ gives $x \ge -2.$

$\therefore -2 \le x \le \dfrac{4}{3}.$

(c) (i) $\tan\theta = \dfrac{x}{\ell}.$

$\theta = \tan^{-1}\dfrac{x}{\ell}.$

$\dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{1}{\ell}\dfrac{1}{1+\frac{x^2}{\ell^2}}\dfrac{dx}{dt} = \dfrac{\ell}{\ell^2 + x^2}\times v = \dfrac{v\ell}{\ell^2 + x^2}.$

(ii) Given that v and ℓ are constant, $\dfrac{d\theta}{dt}$ is the reciprocal of $\ell^2 + x^2$, $\therefore$ It is maximum $\ell^2 + x^2$ is minimum, i.e. when $x = 0$.

$\therefore$ The maximum value of $\dfrac{d\theta}{dt}$ is $\dfrac{v\ell}{\ell^2} = \dfrac{v}{\ell}$.

(iii) $\dfrac{d\theta}{dt} = \dfrac{v}{4\ell}$ gives $\dfrac{v\ell}{\ell^2 + x^2} = \dfrac{v}{4\ell}.$

$4\ell^2 = \ell^2 + x^2.$

$3\ell^2 = x^2.$

$\dfrac{x}{\ell} = \pm\sqrt{3}.$

$\tan\theta = \pm\sqrt{3}.$

$\theta = \pm\dfrac{\pi}{3}.$

Question 4

(a) (i) $T = 190 - 185e^{-kt}$.

When t = 0, T = 190° – 185° = 5°.

$\dfrac{dT}{dt} = 185ke^{-ky} = -k(190 - T)$.

$\therefore$ It satisfies both the equation and the initial condition.

(ii) When t = 1, T = 29: $29 = 190 - 185e^{-k}$.

$185e^{-k} = 161.$

$-k = \ln\dfrac{161}{185} = -0.1390.$

$k = 0.1390.$

When $T = 80, 80 = 190 - 185e^{-0.1390t}$.

$185e^{-0.1390t} = 190 - 80 = 110.$

$-0.1390t = \ln\dfrac{110}{185}.$

$t = \dfrac{\ln\dfrac{110}{185}}{-0.1390} = 3.74.$

3.74 hours = 3 hours 44 minutes.

$\therefore$ The turkey will be cooked at 12:44 pm.

(b) (i) 7! = 5040, (ii) $\dfrac{8!}{2!} = 20160$.

(c) (i) Gradient of QO = $\dfrac{aq^2}{2aq} = \dfrac{q}{2}$.

Gradient of the tangent at P = $\dfrac{2ap}{2a} = p$.

These two lines are perpendicular, $\therefore \dfrac{q}{2}p = -1, \therefore pq = -2$.

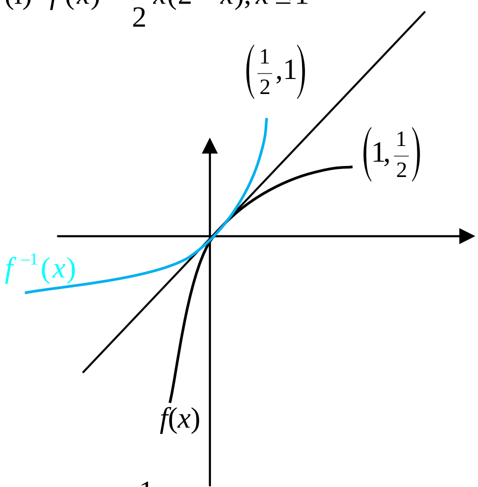
(ii) The gradient of PO is $\dfrac{p}{2}$ and the gradient of the tangent at Q is q , and given $pq = -2$, $\therefore$ PO is perpendicular to the tangent at Q. $\therefore \angle PLQ = 90°$.

(iii) $\angle PLQ = \angle PKQ = 90°$, $\therefore PQLK$ is a semicircle on the diameter PQ.

If M is the midpoint of PQ, M is the centre. $\therefore ML = MK =$ radius.

Question 5

(a) (i) $f(x) = \dfrac{1}{2}x(2-x), x \le 1$



(ii) $f : y = x - \dfrac{1}{2}x^2.$

$f^{-1} : x = y - \dfrac{1}{2}y^2.$

$y^2 - 2y + 2x = 0.$

$(y-1)^2 = 1 - 2x.$

$y = 1 \pm \sqrt{1-2x}$. Take $y = 1 - \sqrt{1-2x}$ so that $y \le 1$.

(c) $f^{-1}\left(\dfrac{3}{8}\right) = 1 - \sqrt{1 - \dfrac{3}{4}} = 1 - \dfrac{1}{2} = \dfrac{1}{2}$.

(b) From $v^2 = n^2(A^2 - x^2)$, when $x = 0, v = 2$, $\therefore 4 = n^2A^2$.

From $a = -n^2x$, when $x = A, |a| = 6, \therefore 6 = n^2A$.

$\dfrac{4}{6} = A, \therefore A = \dfrac{2}{3}$ m.

$4 = n^2\dfrac{4}{9}, \therefore n^2 = 9, \therefore n = 3.$

Period $T = \dfrac{2\pi}{3}$ s.

(c) $\angle PLK = \angle PQM$ (in a cyclic quadrilateral, interior angle = opposite exterior angle)

$\angle PQM = \angle TPM$ (angles in alternate segments are equal)

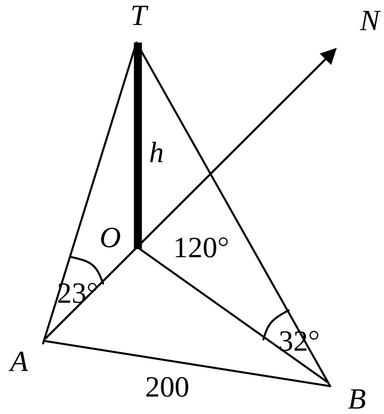
$\angle TPM = \angle LPK$ (vertically opposite angles).

$\therefore \angle PLK = \angle LPK$.

$\therefore \Delta PKL$ is isosceles.

Question 6

(a) (i)



(ii) $\tan 23° = \dfrac{h}{OA}, \therefore OA = h\cot 23°.$

$\tan 32° = \dfrac{h}{OB}, \therefore OB = h\cot 32°.$

$AB^2 = OA^2 + OB^2 - 2OA.OB.\cos(180-120)°.$

$200^2 = h^2\left(\cot^2 23° + \cot^2 32° - \cot 23° \cot 32°\right)$

$h = \dfrac{200}{\sqrt{\cot^2 23° + \cot^2 32° - \cot 23° \cot 32°}} = 96$ m.

(b) $3\sin\theta - 4\sin^3\theta + \sin 2\theta = \sin\theta$.

$2\sin\theta - 4\sin^3\theta + 2\sin\theta\cos\theta = 0.$

$\sin\theta - 2\sin^3\theta + \sin\theta\cos\theta = 0.$

$\sin\theta\left(1 - 2\sin^2\theta + \cos\theta\right) = 0.$

$\sin\theta\left(1 - 2(1-\cos^2\theta) + \cos\theta\right) = 0.$

$\sin\theta\left(2\cos^2\theta + \cos\theta - 1\right) = 0.$

$\sin\theta(\cos\theta+1)(2\cos\theta-1)=0.$

$\sin\theta=0, \cos\theta=-1, \cos\theta=\frac{1}{2}.$

$\therefore \theta=0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi.$

(c) (i)

$(1+x)^{p+q} = {}^{p+q}C_0 + {}^{p+q}C_1 x + {}^{p+q}Cx^2 + \ldots + {}^{p+q}C_{p+q}x^{p+q}.$

$\therefore$ The term independent of x in $\frac{(1+x)^{p+q}}{x^q}$ is ${}^{p+q}C_q$.

(ii) The constant term in the RHS is the sum of the product of each of the following pairs:

$\binom{p}{0}x^0$	$\binom{q}{0}\frac{1}{x^0}$
$\binom{p}{1}x^1$	$\binom{q}{1}\frac{1}{x}$
$\binom{p}{2}x^2$	$\binom{q}{2}\frac{1}{x^2}$
…	…
$\binom{p}{p}x^p$	$\binom{q}{p}\frac{1}{x^p}$

$1+\binom{p}{1}\binom{q}{1}+\binom{p}{2}\binom{q}{2}+\ldots+\binom{p}{p}\binom{q}{p}$ is the coefficient of the constant term in the expansion of $(1+x)^p\left(1+\frac{1}{x}\right)^q$.

$\therefore$ Its simpler expression is ${}^{p+q}C_q$.

Question 7

(a) When $y=h$, $h=Vt\sin\theta-\frac{1}{2}gt^2$.

$gt^2-2V\sin\theta t+2h=0.$

$\sum\alpha=t_1+t_2=\frac{2V\sin\theta}{g}$ and $\prod\alpha=t_1t_2=\frac{2h}{g}.$

(b) $\tan\alpha+\tan\beta=\frac{h}{V\cos\theta}\left(\frac{1}{t_1}+\frac{1}{t_2}\right).$

$=\frac{h}{V\cos\theta}\times\frac{t_1+t_2}{t_1t_2}$

$=\frac{h}{V\cos\theta}\times\frac{\frac{2V\sin\theta}{g}}{\frac{2h}{g}}$

$=\tan\theta.$

(c) $\tan\alpha\tan\beta=\frac{h^2}{V^2\cos^2\theta}\left(\frac{1}{t_1t_2}\right).$

$=\frac{h^2}{V^2\cos^2\theta}\times\frac{g}{2h}$

$=\frac{gh}{2V^2\cos^2\theta}.$

(d) From the diagram,

$r=h\tan\alpha+h\tan\beta=h(\tan\alpha+\tan\beta)$

$w=r-2h\tan\alpha=h(\tan\alpha+\tan\beta)-2h\tan\alpha$

$=h(\tan\alpha-\tan\beta).$

(e) $\tan\phi=\frac{\dot{y}}{\dot{x}}=\frac{V\sin\theta-gt_1}{V\cos\theta}$

$=\tan\theta-\frac{g}{V\cos\theta}\frac{h}{V\cos\theta\tan\alpha}$

$=\tan\theta-\frac{gh}{V^2\cos^2\theta}\frac{1}{\tan\alpha}$

$=\tan\theta-2\tan\beta$

$=\tan\alpha+\tan\beta-2\tan\beta$

$=\tan\alpha-\tan\beta.$

(f) $\frac{w}{r}=\frac{h(\tan\alpha-\tan\beta)}{h(\tan\alpha+\tan\beta)}$

$=\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}=\frac{\tan\phi}{\tan\theta}.$

$\therefore \frac{w}{\tan\phi}=\frac{r}{\tan\theta}.$